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A brief review of E theory

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Abstract

I begin with some memories of Abdus Salam who was my PhD supervisor. After reviewing the theory of non-linear realisations and Kac-Moody algebras, I explain how to construct the non-linear realisation based on the Kac-Moody algebra E_{11} and its vector representation. I explain how this field theory leads to dynamical equations which contain an infinite number of fields defined on a spacetime with an infinite number of coordinates. I then show that these unique dynamical equations, when truncated to low level fields and the usual coordinates of spacetime, lead to precisely the equations of motion of eleven dimensional supergravity theory. By taking different group decompositions of E_{11} we find all the maximal supergravity theories, including the gauged maximal supergravities, and as a result the non-linear realisation should be thought of as a unified theory that is the low energy effective action for type II strings and branes. These results essentially confirm the E_{11} conjecture given many years ago.

0. Memories of Abdus Salam by one of his PhD students

I had the very good fortune to be a PhD student of Abdus Salam, or Professor Salam as us students referred to him. I began in 1973 at Imperial College when supersymmetry was just beginning to be studied after the paper of Wess and Zumino [1]. Abdus Salam was among a small group of largely Europeans who thought that supersymmetry was interesting and had begun working on it in earnest.

For my first year I did not see much of Professor Salam as I was taking my preparatory courses, but once the summer came I had finished my courses and so I went to see Professor Salam to find out what I would work on. To my surprise he asked me what I wanted to do. I said that the infinities in quantum field theory were very ugly and so I would like to work on general relativity which was more aesthetically pleasing. Rather than explain the flaws in this naive approach, he suggested that there was not be so much to do in general relativity and that I might like to look at his very recent paper with John Strathdee in which they had discovered superspace and also super Feynman rules [2]. Within days I was captured by the ideas in this paper and began working on infinities in supersymmetric theories. From the perspective of today I realise that I had been subject to his great charm and diplomacy, a skill which he had used to such great effect all over the world.

Once I had began research I never knew when I would see Professor Salam as he spent most of his time away from Imperial. However, he would come about once a month. The first I knew that he was in the department was when, as I came in to work, I would notice that his door was slightly ajar. I would knock and he would welcome me in. He was always very cheerful, friendly and seemed to have time to talk as if he had all the time in the world. Little did I, as a student, know of the many endeavours for the good of science in the third world that he was undertaking.

At that time the problem was to spontaneously break supersymmetry to find a realistic model of nature that was supersymmetric. Particles in a supersymmetric theory had the same mass and although one could spontaneously break supersymmetry at the classical level [3] [4], the pattern of masses it lead to was not consistent with nature. Professor Salam thought that radiative corrections would break supersymmetry and lead to more promising results. Professor Salam, with John Strathdee, who was in Trieste, produced a series of models and it was my job to compute their one loop effective potentials and see if supersymmetry was spontaneously broken and what pattern of masses they lead to. The first models did not work and as time went on the models became more and more complicated involving very many fields. If I had not completely finished computing with a given model by the time I next meet with Professor Salam it was not a problem, there was always a much better model to look at instead.

In such early days of supersymmetry there were no papers one could look at to get up to speed with the technical difficulties, such a Fierz reshuffles, that were required to work on supersymmetry. Fortunately Professor Salam's long term collaborator Bob Delbourgo had an office nearby and he provided me with all the technical help I needed. We also worked on some of the later models together, swapping rows and columns in matrices of large dimension in order to diagonalise then so as to find the masses, which then turned out to be unsatisfactory.

Eventually I realised that if supersymmetry was preserved at the classical level then

the effective potential vanished in the most general $N = 1$ theory invariant under rigid supersymmetry theory [5]. This meant that one could not spontaneously break supersymmetry using perturbative quantum corrections, although one could still hope that it was spontaneously broken by non-perturbative corrections. The problem of breaking supersymmetry in a natural way is still largely unsolved. The result had another more favourable consequence, as was pointed out by others [6], namely that supersymmetry did solve the hierarchy problem, at least technically. In a supersymmetric version of the standard model the Higgs mass would not be swept up to some large unified scale by quantum corrections as long as supersymmetry was not broken much above the weak scale. This in turn led to the hope that supersymmetry might be found at the LHC.

Talking to Professor Salam you could not escape his great enthusiasm for physics; you came to understand that it was a lot of fun to do physics and that it was good to work in a very relaxed and free thinking way. As became even clearer when I later visited him in Trieste, after I had my PhD, Professor Salam could think of a vast number of ways to proceed in the quest to find new things. He was always most interested in very new ideas and while not all of his ideas worked they included many of the deepest ideas that have come to dominate the subject. As one of his students I was, perhaps, able to absorb some of these qualities. Certainly, it was due to him that I began working on supersymmetry rather than on some uninteresting direction.

I end with an account of three meetings with Abdus Salam that display his warmth and humanity.

At the end of my visit to Trieste the time came for me to leave for the airport. Salam realised that I would be travelling at the same time as the Italian Minister for Science, who was visiting the centre, and so he suggested we could share the same car to the airport. This was met with a frown by the organiser of the visit who, no doubt correctly, thought that a scruffy post-doc with a rucksack might dent the carefully created image that the centre wanted to portray. Of course I went by myself to the airport.

I met Abdus Salam in his office in London a few days after he had won the Nobel prize. I asked him what was it like to win such a prize, he reassured me that he was just the same. He then suggested that we go for coffee in the common room in the old physics building at Imperial. To get there we had to go through a number of doors and he insisted that I go first through each door despite my protests.

During the time that Salam was very ill there was a conference in his honour at Trieste, but he was not well enough to go to all the talks. I saw him sitting at the very back of the big auditorium. I asked if it would be alright to say hello, but I was told that he might not recognise me. Since this might be the last time I would see him I went anyway. I said hello, he put up his hand and I shook it. He then immediately said how was Sue. Sue is my wife's name who he had met only once many years before.

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1. Introduction

Quantum field theory, and in particular quantum electrodynamics (QED), was formulated by Heisenberg and Pauli in 1929-30. However, they realised that there was a problem, all the calculations in this new theory lead to infinities. Indeed, in 1930 Oppenheimer and Waller showed that the self energy of the electron was infinite. The problem arose from the sum over the undetermined momenta circulating in processes involving particles propagating in loops. This led to the feeling that quantum field theory was not a correct theory and that some deeper structures were required. However, just after the end of World War II many distinguished physicists gathered for a meeting at Shelter Island in 1947. Here Kramers pointed out that the final result of the calculation, say for the mass of the electron, was experimentally observed to be finite but the parameters of the theory were not measured and so one could try to absorb the infinities in the parameters. Bethe, on the long train ride back to New York from Shelter Island, used this idea to calculate a quantum field theory (QED) correction to a certain spectral line of hydrogen, the Lamb shift, which had just been experimentally measured. He found the correct result and the way was then clear to calculate more quantities in QED, such as the magnetic moment of the electron, which turned out to agree with subsequent experimental measurements with remarkable precision. Confidence in QED was further boosted when it was shown that the infinities could not only be absorbed for processes involving simple Feynman diagrams but for all calculations in QED and the same for certain other quantum field theories. The 1950-51 papers of Salam were crucial to this result as they solved the problem of overlapping divergences. However, around this time Dyson showed that one could absorb the infinities for only a very limited class of quantum field theories.

However, there were also a significant number of theorists who believed that quantum field theory was not the correct framework to formulate the theory of the weak and strong nuclear forces. In 1937 Wheeler and, independently, in 1943 Heisenberg proposed that one should work with measurable quantities rather than the many non-measurable quantities that appear in quantum field theory and in particular one should study the S-matrix. It was this development that led to string theory.

It is instructive to recall some of the very early developments of particle physics. The first particles to be discovered were the electron, the proton and then neutron (discovered in 1932), then first glimpsed in cosmic rays were the positron (1932), the muon (1936), the pions (1947) and the K mesons (1947). Subsequently the neutrino (predicted by Pauli) was found in 1956 in a nuclear reactor. With the advent of particle accelerators many new particles were found. On the theoretical side, in 1932 Heisenberg suggested that the neutron and proton might form an $SU(2)$ symmetry multiplet if one neglected electromagnetic interactions. Kemmer then used this symmetry to write down an action involving fields for the proton, neutron and the three pions which were a triplet.

The problem for those that wanted to formulate the nuclear weak and strong forces using a quantum field theory was that they had no principle to help them determine the interactions of the many particles which were being discovered. In 1932 Fermi proposed an experimentally successful theory of weak interactions consisting of a four fermion interaction, but it had infinities which could not be absorbed in the parameters of the theory and so it was not consistent with quantum mechanics. In 1935 Yukawa had proposed that there

should exist some massive particles that would mediate the strong nuclear force and when the pions were discovered it was initially thought that they were such particles. However, if one viewed this exchange in the context of a quantum field theory it required a large coupling constant and so it could not be analysed using perturbation theory. Yang-Mills theory was formulated in 1954 and, independently by Shaw, a PhD student of Salam. This did possess a principle that determined interactions and it had the above mentioned $SU(2)$ symmetry in mind, but it was difficult to see at that time how it could be compatible with the observed particles. While the idea that the nuclear weak and strong forces could be mediated by spin one bosons using Yang-Mills theory was being studied there was no consensus on how to do this and what symmetry to use.

A different approach to describe the behaviour of the almost massless pions was put forward. It had been shown, by Goldstone, Salam and Weinberg, that if a quantum field theory possessed a rigid symmetry with group G , that was spontaneously broken to a group H , then one found the dimension of G minus the dimension of H massless particles (Goldstone theorem). Although it was not initially phrased in this way, it was understood that the low energy dynamics of these massless particles was determined by a non-linear realisation of group G with local subgroup H . It was shown that if one took $G = SU(2) \otimes SU(2)$ and $H = SU(2)_{diag}$ then the dynamics predicted by the non-linear realisation agreed with the dynamics of the pions as it was measured in the particle accelerators provided one allowed for their small mass.

The great advantage of using non-linear realisations was that it allowed the pioneers to find some of the symmetries of the theory without having to solve the much more difficult problem of what was the underlying theory, or indeed, even what new conceptual ideas it incorporated. The result was a new appreciation of role of symmetry as a principle to determine interactions and also the importance of spontaneous symmetry breaking. These ideas played a key role in the development of the standard model by Glashow, Salam and Weinberg which is based on the Yang-Mills symmetry $SU(2) \otimes U(1)$ which is spontaneously broken.

Goldstone theorem also played an important role in the development of the standard model. The idea to use spontaneously broken symmetries to give mass to the spin one bosons was thought not to be possible as Goldstone theorem would predict the presence of massless particles that were not observed. The crucial observation was that if a symmetry was a local symmetry then Goldstone's theorem did not apply and one did not find the massless particles predicted by Goldstone's theorem. Thus one could use spontaneous symmetry breaking to mediate the weak nuclear force as long as the symmetry was a local symmetry. Despite this it was not clear if the standard model was consistent as it was known that the use of massive spin one bosons to mediated forces generally lead to infinities of a type that could not be absorbed in the parameters of the theory. However, it was found that if their masses arose from the spontaneous breaking of a local Yang-Mills symmetry then the infinities can be tamed and so the theory was consistent with quantum mechanics.

Perhaps the moral to be drawn from these developments is that the quest to understand nature at its deepest level requires more and more symmetry that is spontaneously broken. Also demanding consistency and mathematical beauty provides a very powerful

guide to finding the correct theory. The standard model emerged from combining special relativity and quantum theory in a way that lead to consistent theory, and in particular, one whose predictions were free from infinities. It is also clear that progress on such a difficult problem had to proceed step by step and there was no hope that anyone could guess the final theory, or even the main underlying concepts in one single leap.

The history of particle physics I have given has been partly derived from the books and papers in reference [0]. However, not having studied most of the original papers I may not have the emphasis correct in some places.

The quantisation of Einstein's theory also lead to the infinities mentioned at the beginning of this introduction, but unlike for QED and the standard model, these can not be absorbed in the parameters of the theory. This can be seen to be a consequence of the fact that the vertices of this theory, unlike that of Yang-Mills theory contain momentum squared factors and this in turn can be traced to the requirement that the theory be invariant under general coordinate transformations. These transformations contain a parameter which always comes with an associated derivative unlike the transformations of the Yang-Mills gauge field. Thus it appears that the application of quantum theory and Einstein's theory of general relativity in the context of point particle quantum field theory does not lead to a consistent theory of quantum gravity.

As we have mentioned, the disillusionment with point particle quantum field theory lead to S-matrix theory which in turn lead to string theory. The all loop scattering amplitudes of the bosonic string were computed at a very early stage of string theory, although the integrand over the moduli of the Riemann surface was not evaluated in general [1,2]. These calculations lacked the contribution due to the ghosts as their necessity was not understood at this time, but this was provided later and resulted in only slight changes to the amplitudes. The results can be written in a very compact way and one can think that the simplest things about string theory are the amplitudes themselves. String theory does provide a consistent theory of quantum gravity when viewed from a perturbative perspective in that the amplitudes are essentially free from the infinities referred to above. The price is that one has an infinite number of particles corresponding to the vibrational modes that exist on the string length. However, there is no truly systematic way to compute non-perturbative effects in string theory and from this viewpoint string theory is not complete.

One finds that closed strings contain a graviton and that the open strings contain gauge particles. Indeed, at low energy, open strings describe contain Yang-Mills gauge theory [3] while closed strings contain Einstein's theory [4]. Thus string theory has the potential to contain the particles responsible for the forces that we know.

The type II superstrings are, by definition, those that possess a spacetime supersymmetry with a parameter that has thirty two components. There are two such theories called IIA and IIB. The massless particles of the superstrings are essentially determined by this supersymmetry and they are of necessity the states of the type II supergravity theories in ten dimensions which by definition have the same number of supersymmetries. The low energy actions of the superstrings are by definition theories whose degrees of freedom are these massless particles. These theories are complete in that they contain all effects at low energy, including all the effects that are due to the heavier particles in intermediate

processes. One can think of them as arising once the integral over the heavier particles has been carried out in the Feynman path integral of the underlying theory.

Given the very powerful character of spacetime supersymmetry in determining invariant actions, or equations of motion, it is natural to believe that the low energy effective action of the type IIA superstring is the type IIA supergravity [5] and the type IIB superstring is the type IIB supergravity [6,7,8], indeed this was the motivation for the construction of these supergravity theories. One might expect that these supergravity theories should contain all the perturbative and non-perturbative superstring effects at low energy. As such they have provided one source of knowledge about string theory that is complete and many developments have arisen from thinking about superstring theory from this perspective. In particular these supergravity theories contain solutions that correspond to strings but also branes. An obvious anomaly was the existence of the eleven dimensional supergravity theory [9]. This theory does not possess a string solution but rather a two brane solution. The IIA and IIB supergravity theories also contained brane solutions and one is lead to the expectation that strings and branes should be treated on a more equal footing.

One of the most unexpected developments in supersymmetric theories was that supergravity theories possesses unexpected symmetries. Indeed the maximal supergravity theory in four dimensions was found to possess an E_7 symmetry [10]. More generally the maximal supergravity in D dimensions were found to have an E_{11-D} symmetry [11]. These studies did not include the IIB supergravity theory that was found to have a $SL(2,R)$ symmetry [6]. These symmetries are associated with the scalar fields that these theories possess and indeed the dynamics of the scalar fields in these theories are described by a non-linear realisation of these groups. Such symmetries are broken by the presence of solitons and the quantised charges they possess and it was first proposed in the context of the heterotic string that such symmetries, when suitably discretised, might be symmetries of string theory [12]. This was generalised to the type II superstrings and in particular the conjecture that the IIB superstring should have a $SL(2, Z)$ symmetry [13]. One very interesting consequence of these symmetries was that they transformed the string coupling in such a way that they took the weak (small) coupling regime to the string (large) coupling regime of these theories [12,13].

The type II supergravity theories are connected by a number of relations. The dimensional reduction of the eleven dimensional supergravity theory on a circle leads to the IIA supergravity theory in ten dimensions, indeed this was how this latter theory was constructed. We note that dimensional reduction on a circle preserves the number of supersymmetries. Further dimensional reduction leads to the unique maximal supergravity theories on nine and less dimensions. By a maximal supergravity we mean a theory that is invariant under a supersymmetry that has thirty two component parameters. The dimensional reduction of the IIB supergravity theory leads to the same nine dimensional theory; this must be the case as the nine dimensional maximal supergravity theory is unique. Thus there is a mapping between the IIA and IIB supergravity theories on a circle. These relations inherit into corresponding superstring theories. The strong coupling limit of the IIA string theory can be thought of as defining an eleven dimensional theory whose low energy limit is the eleven dimensional supergravity theory [14]. The relations between the

IIA and IIB superstring theories on circles is an example T duality transformation.

These ideas have come to be known as M theory but, as is rather clear, this is not a theory rather it is a set of relations between the different theories.

The problem with including branes as elements in the underlying theory is that unlike the string there is very little known about how to quantise branes. While there has been progress on formulating an action for multiple coincident M2 branes and the use of open strings has allowed us to understand some properties of D branes one does not even know the quantum states of a single brane and the problem of scattering amplitudes for branes is still very far from being solved. Thus progress in string theory has lead us into some kind of no mans land in that we realise that we understand very little about what the underlying theory could be.

In this review we will explain that all the maximal supergravity theories, that is, the low energy effects of the superstring theories, can be unified in a single theory that contains a very large new symmetry, the Kac-moody algebra E_{11} . This theory is a non-linear realisation of the semi-direct product of E_{11} with its vector representation denoted $E_{11} \otimes_s l_1$. This field theory is similar to that used to formulate the dynamics of pions mentioned above, but it differs in that it automatically contains a spacetime as part of the group structure. This is a first step that one can hope may be used to determine some of the properties of the underlying theory of string and branes.

We begin by giving a review of the theory of non-linear realisations in section two. Section three contains a short account of Kac-Moody algebras followed by section four gives some of the historical motivation for the idea that the underlying theory of strings and branes has an E_{11} symmetry. Section five constructs the Kac-Moody algebra E_{11} and its representation which of most interest to us, the l_1 , or vector, representation. Section six contains the construction of the non-linear realisation of $E_{11} \otimes_s l_1$ non-linear realisation and the eleven dimensional dynamics it predicts. Section seven explains how the theories in D dimensions emerge from this non-linear realisation and section eight shows that the non-linear realisation of $E_{11} \otimes_s l_1$ is a unifying theory in that it contains the many different type II maximally supersymmetric theories. Section nine is a discussion of the meaning of the results. This review is an expanded version of the lecture given in Singapore.

2. Non-linear realisations

As mentioned in the introduction the theory of non-linear realisations was once well known but this knowledge has largely been lost, and worse still, been replaced by misunderstandings. As a result in this section we will review the theory of non-linear realisation. The data required to specify a non-linear realisations is a group G with a choice of subgroup H . The non-linear realisation of a group G with local subgroup H is, by definition, constructed out of a group element $g \in G$ which is subject to the transformations

$$g \rightarrow g_0 g, \quad g_0 \in G, \quad \text{as well as} \quad g \rightarrow gh, \quad h \in H \quad (2.1)$$

The group element $g_0 \in G$ is a rigid transformation, that is, it is a constant, while $h \in H$ is a local transformation, that is, like g it depends on the space-time that the theory possess. The spacetime may be introduced by hand, as was the case for the original use of the non-linear realisation used in particle physics, or it may be introduced as part of the

construction by including corresponding generators that belong to the group G . This latter case is the one of interest to us in this paper. Clearly we can use the local transformation h to gauge away part of the group element g .

We now take the group G to have a particular form, that is, it is the semi-direct product of a group \hat{G} with one of its representations l ; we denote this by $\hat{G} \otimes_s l$. We denote the generators of \hat{G} by R^α and for each element of the l representation we introduce a corresponding generators l_A . Then the algebra $\hat{G} \otimes_s l$ can be written as

$$[R^\alpha, R^\beta] = f^{\alpha\beta}{}_\gamma R^\gamma \quad (2.2)$$

$$[R^\alpha, l_A] = -(D^\alpha)_A{}^B l_B \quad (2.3)$$

The first equation is just the Lie algebra for \hat{G} and in the second equation the matrix $(D^\alpha)_A{}^B$ is the matrix representation of the l representation. One can verify that it satisfies the Jacobi identity by virtue of this fact. The commutators of the l generators are restricted by the Jacobi identity. The simplest consistent choice is to take them to commute but we will leave them unspecified for the time being.

The reader is very familiar with the notion of the semi-direct product as the Poincare group P in D dimensions can be written as $P = SO(1, D-1) \otimes_s T^D$ where T^D are the translations generators corresponding to the vector representation of $SO(1, D-1)$. If we denote the spacetime translations by P_a and the Lorentz rotations by J_{ab} , with $a, b, \dots = 0, 1, 2, \dots, D-1$ then the Lorentz algebra is given by

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc} \quad (2.4)$$

while equation (2.3) is written, for this case, as

$$[J_{ab}, P_c] = -\eta_{ac} P_b + \eta_{bc} P_a \quad (2.5)$$

The group element g of $\hat{G} \otimes_s l$ can be written in the form

$$g = e^{x^A l_A} e^{A_\alpha R^\alpha} \equiv g_l g_A \quad (2.6)$$

where x^A and A^α parameterise the group element and in the second equation g_A and g_l involves the generators of \hat{G} and the l representation respectively. We will interpret the x^A as the coordinates of a spacetime and the A^α as fields that live on this spacetime, that is, they depend on the coordinates x^A .

The dynamics of the non-linear realisations is just a set of equations that are invariant under the transformations of equation (2.1). To understand why the non-linear realisation leads to equations of motion one just has to realise that the group element g of equation (2.1) contains the fields of the theory which depend on the generalised space-time. As a result when one finds a set of quantities, constructed out of the group element g , that is, invariant under the transformations of equation (2.1) one is necessarily constructing an equation of motion for the fields of the theory. Hence the non-linear realisation leads to dynamical equations for the fields which are either unique, or almost unique, provided one specifies the number of derivatives involved. As with every application of any symmetry

one has to specify the number of spacetime derivatives the action should contain. Non-linear realisations are a bit different to the more familiar situation where one has some fields that transform linearly under a symmetry as in the case of the non-linear realisation the symmetry and the fields are very closely linked and it is this that leads to the prediction of the dynamics in such a precise way.

We now consider three types of non-linear realisation, one that leads just to a spacetime, one that leads to fields that depend on a spacetime that is introduced by hand and finally one that leads to a spacetime and fields that depend on this spacetime. We denote these as types I, II and III.

1. Type I

Let us first consider the case that the local subgroup $H = \hat{G}$ and in this case we can write the group element in the form $g = e^{x^A l_A}$ as the second factor in the group element involving the group \hat{G} can be gauged away using the local H transformation of equation (2.1). Thus in this case we are just left with the coordinates x^A and there are no fields. If we take G to be the Poincare group, that is, $\hat{G} = SO(1, D-1)$ and $H = SO(1, D-1)$ the group element is $g = e^{x^a P_a}$ and the transformations resulting from the rigid transformation are the Poincare transformations of Minkowski spacetime.

Another classic example is to take G to be the super Poincare group in four dimensions with one supercharge Q_α and $H = SO(1, D-1)$. The super Poincare group is a semi-direct product of the Lorentz group and its representation consisting of P_a and Q_α . We note that in this case the elements of the albeit reducible l representation no longer commute with themselves. The group element can be chosen to be of the form $g = e^{x^a P_a} e^{\theta^\alpha Q_\alpha}$ and the rigid transformations are those of superspace first found in the classic paper of Salam and Strathdee, reference [2] in the first section.

Type I non-linear realisation contain no fields and are just the cosets G/H found in elementary mathematics books on group theory.

The Cartan forms for this type of non-linear realisation are given by

$$\mathcal{V} = g^{-1} dg = dx^\Pi e_\Pi^A l_A + dx^\Pi \omega_{\Pi, \alpha} R^\alpha \quad (2.7)$$

By studying the transformations of the objects e_Π^A and $\omega_{\Pi, \alpha}$ under the non-linear realisation one finds that they can be taken to be the vielbein and spin connection of the coset space G/H . It is this interpretation that encourages the use of the local indices Π , .. rather than the tangent indices A, \dots according to whether they transform under local H transformations or rigid g_0 transformations induced from such transformations on the coordinates. The tangent space of the coset has tangent group H and it is easy to find that e_Π^A and ω_Π^α transform under the local group H as they should.

2. Type II

We now consider a second kind of non-linear realisation which involves taking no generators in the l representation, that is $G = \hat{G}$ and H is a subgroup of G . The group element takes the form $g = e^{A_\alpha R^\alpha}$. So far we have no spacetime but, by hand, we introduce a spacetime with coordinates x_A simply by taking the fields A_α to depend on these coordinates. We note that the coordinates are dummy variables and, in this case, have no

relation with the generators G . Local in this case means that the group element g and the local transformations h of equation (2.1) depend on the coordinates x_A . One can use the local symmetry to choose the group element to be of a particular form and so set to zero some of the fields A_α . Indeed the number of fields one can set to zero is the dimension of H leaving the dimension of G minus the dimension of H fields. A fact that is consistent with Goldstone's theorem.

Our problem is to find the dynamics that is invariant under the transformations of equation (2.1). The usual method is to construct the Cartan forms

$$\mathcal{V} = g^{-1}dg = P + Q \quad (2.8)$$

where Q belongs to the Lie algebra of H and P contains only the remaining generators in G . Considering the rigid transformations of equation (2.1) we see that the Cartan forms are invariant under these transformations. However, under the local transformations they transform as

$$\mathcal{V}' = h^{-1}\mathcal{V}h + h^{-1}dh \quad (2.9)$$

Clearly the P part of the Cartan form transforms covariantly, that is, as $P' = h^{-1}Ph$. If we demand that the action we seek has only two derivatives then it is of the form

$$\int dx_A \text{Tr}(P^2) \quad (2.10)$$

where we have chosen the generators of G to be in a particular representation, indeed any matrix representation will do. The number of possible terms in the action one can write is determined by the way the adjoint representation of G decomposes into the representations of H . If there is only one representation in addition to the adjoint representation of H then the action is unique. The general theory for this type of non-linear realisation was given in the classic papers [15]. A more extensive review of this type of non-linear realisation can be found in section 13.2 of reference [16].

As we have mentioned above the maximal supergravity theories have some symmetries associated with the scalars. In fact the dynamics of the scalars in these supergravity theories is just the non-linear realisation of the corresponding symmetry group. In particular the IIB supergravity theory has two scalars and their dynamics is the non-linear realisation of $SL(2, \mathbb{R})$ with local subgroup $SO(2)$ [6], while the maximal supergravity theory in four dimensions has 70 scalars that belong to the non-linear realisation of E_7 with local subgroup $SU(8)$ [10]. In general the scalars in the maximal supergravity theory in D dimensions, for $D \leq 9$, belong to the non-linear realisation E_{11-D} with a local subgroup which is the maximal compact subgroup of E_{11-D} .

It was a type II non-linear realisation that was used to account for the pion dynamics, discussed in the introduction, by taking $G = SU(2) \otimes SU(2)$ and H to be the diagonal $SU(2)$ subgroup. The low energy dynamics is uniquely determined.

2. Type III

Finally we give an account of the type of non-linear realisation used in this talk. Now we consider no restriction and so $G = \hat{G} \otimes_s l$ and the local subgroup H is a subgroup of \hat{G} .

The group element has the form of equation (2.6) and we find a spacetime with coordinates which are in one to one correspondence with generators in the l representation. The fields A_α depend on the coordinates x_A and the group element transforms as in equation (2.1).

Since the generators in the l representation belong, by definition, to a representation of \hat{G} we can write the transformations of equation (2.1) under the rigid g_0 belonging to \hat{G} act as

$$g'_l = g_0 g_l g_0^{-1}, \quad g'_A = g_0 g_A \quad (2.11)$$

An exception is when the rigid transformations $g_0 \in l$ and in this case they just give a shift the coordinates. While the local $h \in H$ transformations act as

$$g'_l = g_l, \quad g'_A = g_A h \quad (2.12)$$

As a result the local subalgebra transformations only change the fields and leave the coordinates alone.

To construct the dynamics we consider the Cartan forms which now take the form

$$\mathcal{V} \equiv g^{-1} dg = \mathcal{V}_A + \mathcal{V}_l, \quad (2.13)$$

where

$$\mathcal{V}_A = g_A^{-1} dg_A \equiv dz^\Pi G_{\Pi, \alpha} R^\alpha, \quad (2.14)$$

belongs to the Lie algebra \hat{G} and are the Cartan forms for \hat{G} , while the part that contains the generators of the l representation is given by

$$\mathcal{V}_l = g_E^{-1} (g_l^{-1} dg_l) g_E = g_E^{-1} dz \cdot l g_E \equiv dz^\Pi E_\Pi^A l_A \quad (2.15)$$

While both \mathcal{V}_E and \mathcal{V}_l are invariant under rigid transformations, under local transformations of equation (2.5) they transform as the

$$\mathcal{V}_E \rightarrow h^{-1} \mathcal{V}_E h + h^{-1} dh \quad \text{and} \quad \mathcal{V}_l \rightarrow h^{-1} \mathcal{V}_l h \quad (2.16)$$

Type III non-linear realisations were not as well studied as type II in the old days. However, Isham, Salam and Strathdee worked out in detail the non-linear realisation of the conformal group in four dimensions with the local subgroup being the Lorentz group [17]. Borisov and Ogivestsky considered the non-linear realisation of $GL(4) \otimes_s T^4$ [18]. In this case the dynamics was not unique but one could choose the undetermined coefficients so that it lead Einstein's gravity. A review of this calculation in D dimensions can be found in section 16.2 of reference [16] which also develops the theory of type III non-linear realisation further as was done in the E_{11} papers referenced later on in this review. An early review which also contains a discussion of these type III non-linear realisations can be found in reference [19].

3. Kac-Moody algebras

In this section we will explain how Kac-Moody algebras were discovered [20, 21] and by doing so give some insight into what they are. We will gloss over many important

points, however, the reader can read a detailed and pedagogical account of Kac-Moody algebras in chapter 16 of reference [16]. Group theory emerged from the study of the roots of polynomial equations, however, physicists are more used to thinking of groups as sets of matrices. Given a group it was found that one could reconstruct the part connected to the identity by considering the Lie algebra. It also turns out that all finite dimensional Lie algebras can be constructed from a subset of Lie algebras that are finite dimensional and semi-simple and so we work just with these. The precise meaning of semi-simple can be found in section 16.1 of reference [16]. The Lie algebra, denoted E , contains a set of commuting generators which we denote by H_i , $i = 1, 2, \dots, r$ where r is by definition the rank of the algebra. This Abelian algebra is called the Cartan subalgebra. We can now diagonalise the remaining generators E_α with respect to the Cartan subalgebra, that is, we write the commutators of all the other generators in the Lie algebra E with those of the Cartan subalgebra generators in the form $[H_i, E_\alpha] = \alpha_i E_\alpha$. In doing this we find a set of vectors α_i called the roots. A basis for the roots is called the simple roots, and we denoted them as α_a , $a = 1, 2, \dots, r$. Given the simple roots we can construct their scalar products to form the Cartan matrix which is defined by

$$A_{ab} = \frac{2(\alpha_a, \alpha_b)}{(\alpha_a, \alpha_a)} \quad (3.1)$$

The classification of Lie algebras is usually carried out when the Lie algebra E is considered to be over the complex numbers. However, it turns out that the Cartan matrix is real and has integer values. The Dynkin diagram consists of r dots that are connected by a set of lines which are drawn according to the Cartan matrix using a set of rules. The rules are such that given the Dynkin diagram one can deduce the Cartan matrix uniquely.

Killing, together with later work by Cartan, found that all the Lie algebras they knew lead to Cartan matrices with the properties

$$A_{ab} \leq 0 \text{ if } a \neq b \quad (3.2)$$

$$\text{if } A_{ab} = 0 \text{ then } A_{ba} = 0 \quad (3.3)$$

$$v^a A_{ab} v^b \geq 0 \text{ for any real vector } v^a \quad (3.4)$$

By construction $A_{aa} = 2$ and the positive definite nature of the Cartan matrix implies that the off diagonal entries can only take the values $0, -1, -2, -3$

Killing looked at the possible list of Cartan matrices that satisfied the above properties and he found that there were some that did not corresponding to any Lie algebra that he knew. By finding the Lie algebras that lead to these new Cartan matrices he discovered some new algebras which were the exceptional algebras F_4, G_2, E_6, E_7 and E_8 .

In the above discussion we started from a Lie algebra and found a Cartan matrix. However, in the 1950's Serre showed that one could go the other way around, that is, start from the Cartan matrix and reconstruct the corresponding Lie algebra. He introduced $3r$ generators E_a, F_a and H_a . The Lie algebra was just given by all commutators of these generators subject to certain relations between these commutators that are completely specified by the Cartan matrix. We will not give them here but they can be found in

section 16.1 of reference [16]. Once the Lie algebra has been constructed one can identify the H_a as the Cartan subalgebra generators in a different basis, the E_a as the generators corresponding to the simple roots α_a and the F_a as the generators corresponding to the roots $-\alpha_a$.

Kac-Moody algebras were discovered in 1969 [20, 21]. As Serre advocated we start from a Cartan matrix and construct the Lie algebra in the same way and subject to the same relations. However, we now allow Cartan matrices that obey only equations (3.2) and (3.3) but not necessarily equation (3.4).

Clearly if equation (3.4) holds then the Kac-Moody algebra constructed is one in the Cartan-Killing list of Lie algebras, that is, the list of finite dimensional semi-simple Lie algebras. When the Cartan matrix is positive semi-definite with only one zero eigenvalue one finds that the construction leads to the already known and well understood affine Lie algebras. However in general one finds a vast new class of algebras whose properties are largely unknown. In particular one does not know a listing of the generators for even one of these new Kac-Moody algebras.

4. Historical motivation for an E_{11} symmetry

As we have mentioned already, the E_7 symmetry of the maximal supergravity theory in four dimensions is associated with the seventy scalars whose dynamics is just a type II non-linear realisation, discussed in section two, for the group E_7 with local subgroup $SU(8)$ [10]. As discussed in section two we may use the local symmetry to choose part of the group element g used in the non-linear realisation. In particular we may use the local subgroup $SU(8)$ to remove part of the group element g and it turns out that one can choose the group element to belong to the Borel subalgebra of E_7 . This is consistent with the count $133 - 63 = 70$. As a result every scalar in the supergravity theory arises in the non-linear realisation from a generator in the Borel subgroup of E_7 . Indeed this is the general pattern for the maximal supergravity theories, one can use the local transformations in D dimensions to choose the group element associated with the non-linear realisation, to which the scalars belong, to be in the Borel subalgebra of the symmetry group E_{11-D} . This is related to the fact that the local subgroups used in the non-linear realisations are the maximally compact subgroups of E_{11-D} , or more technically the Cartan Involution invariant subgroup.

The E_{11-D} symmetries that arise in D dimensional maximal supergravity theories were universally thought to be a quirk of the dimensional reduction procedures used to obtain these theories. However, it was shown that the eleven dimensional supergravity theory was a nonlinear realisation [22]. This theory has no scalars but by introducing the generators K^a_b , $R^{a_1 a_2 a_3}$ and $R^{a_1 \dots a_6}$ corresponding to the graviton h_a^b , three form $A_{a_1 a_2 a_3}$ and six form $A_{a_1 \dots a_6}$ fields respectively, and taking them to obey a suitable algebra \mathcal{A}^{11} , one could construct a non-linear realisation that lead to the eleven dimensional supergravity theory. We note that these generators carry indices that transform under the spacetime transformations, unlike for the non-linear realisations that occurs for the scalar fields. Not every theory can be formulated as a non-linear realisation and so this result told us something about eleven dimensional supergravity. However, the dynamics of this non-linear realisation was not unique and it contained some constants that had to be fixed by hand to the required values.

One motivation for this construction was the previously mentioned, and rather old, result of Borisov and Ogievetsky [18] which showed that gravity in four dimensions could be formulated as a (type III) non-linear realisation of $GL(4) \otimes_s l$ with local subgroup $SO(1,3)$ where l is the vector representation of $SL(4)$. As we have just mentioned the gravity sector in the eleven dimensional supergravity theory arose from the $K^a{}_b$ generators which belong to the algebra $GL(11)$.

The algebra \mathcal{A}^{11} that emerged from formulating the eleven dimensional supergravity theory as a non-linear realisation was not a finite dimensional algebra in the list of Cartan and nor was it a Kac-Moody algebra. This was to be expected as using this construction one only finds generators associated with the fields of the theory and this does not include generators for any local subalgebra. Indeed one should expect to find only the Borel subalgebra of some large algebra. However, if one demanded that this algebra \mathcal{A}^{11} was contained in a Kac-Moody algebra then the smallest such algebra was E_{11} [23]. Motivated by this realisation it was conjectured that the E_{11-D} symmetries were not a quirk of dimensional reduction but that the exceptional symmetries found in the lower dimensional maximal supergravity theories were part of a vast E_{11} symmetry of the eleven dimensional theory [23]. The price to pay for changing the algebra used in the non-linear realisation to E_{11} was that it lead to a theory that contains an infinite number of fields, only the first few of which were those of eleven dimensional supergravity.

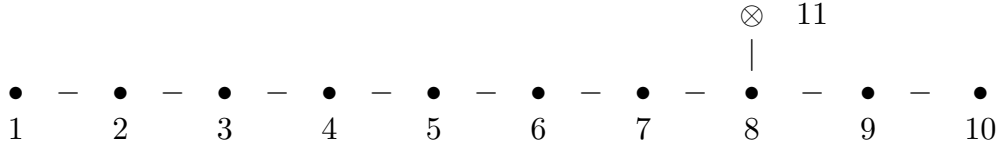
It is instructive to examine how the above construction would have proceed for the four dimensional maximal supergravity. This theory can also be formulated as a non-linear realisation. To do this one introduces for each field of the theory a generator and adopts a suitable Lie algebra that they satisfy which can largely be found by requiring that the corresponding non-linear realisation gives the equations of motion of four dimensional maximal supergravity theory. In particular for the scalar fields one introduces generators, which carry no Lorentz indices, in the Borel subalgebra of E_7 , however, we must also introduce the generators $K^a{}_b$ for the graviton and R^{aN} for the vectors. We can think of this as extending the non-linear realisation of the scalars to include the other fields and in so doing so we must introduce generators that transform non-trivially under spacetime transformations. We note that we only have some of the generators of the full algebra in particular we have only the Borel subalgebra of E_7 rather than the full E_7 algebra. Demanding that the algebra be extended to a Kac-Moody algebra leads to the E_7 algebra in the scalars sector, but the algebra E_{11} for the full theory.

In the above we have sidestepped the question of how we are to introduce spacetime into the theory. Thinking about the gravity sector of the non-linear realisation it is apparent that we should consider a type III non-linear realisation, that is, include generators in the algebra which lead to the coordinates of spacetime, rather than the type II non-linear realisation used for the scalars. In the first papers on E_{11} one just introduced the space time translation generators P_a even though it was clear that this could only be part of the solution as it was not an E_{11} covariant introduction. The correct way to introduce spacetime is to introduce generators corresponding to a representation of E_{11} , the l_1 representation, which generalises the spacetime translations to include an E_{11} multiplet of generators, and take the algebra used in the non-linear realisation to be the semi-direct product of E_{11} and this l_1 representation [24].

The above method of proceeding has an analogy with the original use of non-linear realisations in particle physics. The analogue of pion dynamics is the maximal supergravity theories as these are thought to contain the low energy dynamics of strings and branes. The algebra $SU(2) \otimes SU(2)$ is replaced with E_{11} . Of course the theory of pion dynamics was finally understood after the introduction of quarks and the $SU(3)$ gluon gauge theory. However, as we explained in the introduction this approach to pion dynamics played a key role in unravelling the correct theory, that is, the standard model.

5 The E_{11} algebra, its vector representation and the $E_{11} \otimes_s l_1$ algebra

The Dynkin diagram of the Kac-Moody algebra E_{11} is given by



As with any Kac-Moody algebra we do not know all the generators of E_{11} . However, one can gain some understanding of the E_{11} algebra by considering the decomposition of E_{11} into representations of $SL(11)$. In the Dynkin diagram this corresponding to deleting node eleven so as to leave the algebra $SL(11)$. This is the meaning of the cross in the E_{11} Dynkin diagram. The decomposition leads to a set of generators which belong to representations of $SL(11)$ and can be classified according to a level. The level is the number of the number of up minus down $SL(11)$ indices divided by three and it is preserved by the commutators of E_{11} . We will denote the generators of E_{11} by R^α and those of the l_1 representation by l_A . The positive level generators are given by [23]

$$K^a_b, R^{a_1 a_2 a_3}, R^{a_1 a_2 \dots a_6} \text{ and } R^{a_1 a_2 \dots a_8, b}, \dots \quad (5.1)$$

where the generators $R^{a_1 a_2 a_3}$ and $R^{a_1 a_2 \dots a_6}$ are totally antisymmetric in their indices, while the next generator obey the constraints $R^{a_1 a_2 \dots a_8, b} = R^{[a_1 a_2 \dots a_8], b}$ and $R^{[a_1 a_2 \dots a_8, b]} = 0$. The indices $a, b, \dots = 1, 2 \dots 11$. The generators K^a_b are those of $GL(11)$ and have level zero, the other generators have levels 1, 2, 3, The negative level generators are given by

$$R_{a_1 a_2 a_3}, R_{a_1 a_2 \dots a_6}, R_{a_1 a_2 \dots a_8, b}, \dots \quad (5.2)$$

where the last generator obey an analogous constraint.

The E_{11} algebra can be constructed using the Serre procedure but it is much easier to construct it level by level using the known generators at a given level, the fact that the commutators preserve the level and obey the Jacobi identity. The $GL(11)$ generators, by definition, obey the commutators

$$[K^a_b, K^c_d] = \delta^b_c K^a_d - \delta^a_d K^b_c \quad (5.3)$$

The level one commutators are those between the generators of $SL(11)$ and the generators $R^{a_1 a_2 a_3}$ and are given by

$$[K^a_b, R^{c_1 c_2 c_3}] = \delta^{c_1}_b R^{a c_2 c_3} + \delta^{c_2}_b R^{c_1 a c_3} + \delta^{c_3}_b R^{c_1 c_2 a} \quad (5.4)$$

It just expresses the fact that the generators belong to a representation of $SL(11)$.

At the next level we find the commutator

$$[R^{a_1 a_2 a_3}, R^{b_1 b_2 b_3}] = 2R^{a_1 a_2 a_3 b_1 b_2 b_3} \quad (5.5)$$

This latter result is obvious given that $R^{a_1 a_2 a_3}$ has level one and so the commutator of two of them must have level two and must be equal to the only generator at that level. The choice of factor of 2 fixes the normalisation of the level two generator in the algebra. The E_{11} algebra is known up to level three [23]. The reader can find a detailed account of its construction and the result in chapter 16 of reference [16].

The first fundamental representation of E_{11} , denoted the l_1 representation is the representation with highest weight Λ_1 which obeys the equation $(\Lambda_1, \alpha_a) = \delta_{1,a}$. We will also refer to this representation the vector representation of E_{11} . As for the E_{11} algebra we also consider the representation when decomposed into representations of $SL(11)$. It can be constructed using the standard techniques involving raising and lowering generators. One finds that the vector representations contains the elements [24]

$$\begin{aligned} &P_a, Z^{ab}, Z^{a_1 \dots a_5}, Z^{a_1 \dots a_7, b}, Z^{a_1 \dots a_8}, Z^{b_1 b_2 b_3, a_1 \dots a_8}, Z^{(cd), a_1 \dots a_9}, \\ &Z^{cd, a_1 \dots a_9}, Z^{c, a_1 \dots a_{10}} (2), Z^{a_1 \dots a_{11}}, Z^{c, d_1 \dots d_4, a_1 \dots a_9}, Z^{c_1 \dots c_6, a_1 \dots a_8}, Z^{c_1 \dots c_5, a_1 \dots a_9}, \\ &Z^{d_1, c_1 c_2 c_3, a_1 \dots a_{10}}, (2), Z^{c_1 \dots c_4, a_1 \dots a_{10}}, (2), Z^{(c_1 c_2, c_3), a_1 \dots a_{11}}, Z^{c, a_1 a_2}, (2), Z^{c_1 \dots c_3, a_1 \dots a_{11}}, (3), \dots \end{aligned} \quad (5.6)$$

The blocks of indices contain indices that are totally antisymmetrised while $()$ indicates that the indices are symmetrised. All the elements come with multiplicity one except when there is a bracket which gives the multiplicity. All the generator belong to irreducible representations of $SL(11)$, for example $Z^{a_1 \dots a_7, b}$ obeys the constraint $Z^{[a_1 \dots a_7, b]} = 0$. The l_1 generators are also classified by a level which is the number of up minus down indices plus one. We have listed the generators up to and including level five.

The level zero entry follows from the observation that, at level zero, we delete node eleven in the E_{11} Dynkin diagram leaving the $SL(11)$ algebra and so we have the first fundamental representation of $SL(11)$ which is a vector of $SL(11)$, that is, P_a . We note that the first three entries have the same form as the central charges of the eleven dimensional supersymmetry algebra, but these are only a small part of the vector representation. In fact E_{11} seems to systematically predict results which are usually considered to follow from supersymmetry.

The charges for the point particle, the two brane and five brane are the first three objects respectively in the l_1 representation. There is very good evidence that the l_1 representation contains all branes charges [24,25,26,27]

To construct the algebra $E_{11} \otimes_s l_1$ we promote the elements of the vector representation to be generators and then find the commutators of equation (2.3) for this case. The simplest way is to proceed level by level preserving the level and implementing the Jacobi identity. One finds for the first two levels that [24]

$$[K^a_b, P_c] = -\delta^a_c P_b + \frac{1}{2} \delta^a_b P_c \quad (5.7)$$

$$[R^{a_1 a_2 a_3}, P_b] = 3\delta_b^{[a_1} Z^{a_2 a_3]} \quad (5.8)$$

We take the l_1 generators to commute. The $E_{11} \otimes_s l_1$ algebra is known up to level four and can be found up to level three in chapter 16 of [16] where a detailed account of this algebra and its construction can be found. This includes the perhaps unexpected extra term in equation (5.7).

The lists of E_{11} and l_1 generators to quite high levels can be found in reference [16] and by using the Nutma programme Simplie [28].

6. The non-linear realisation of $E_{11} \otimes_s l_1$

To construct this non-linear realisation $E_{11} \otimes_s l_1$ we follow the procedure given in section two for a non-linear realisation of type III. We need to know not only the algebra $E_{11} \otimes_s l_1$, discussed in the last section, but also the local subalgebra. Any Kac-Moody algebra possess an involution, called the Cartan involution. This takes the generators associated with positive roots into generators associated with the corresponding negative roots. We take the local subalgebra of the non-linear realisation $E_{11} \otimes_s l_1$ to be the subalgebra of E_{11} that is invariant under this Cartan involution. We denote this algebra by $I_c(E_{11})$ and it contains the generators

$$I_c(E_{11}) = \{K^a{}_b - \eta^{ac}\eta_{bd}K^d{}_c, R^{a_1 a_2 a_3} - \eta^{a_1 b_1}\eta^{a_2 b_2}\eta^{a_3 b_3}R_{b_1 b_2 b_3}, \dots\} \quad (6.1)$$

We notice that it does, as expected, involve generators that are a sum of positive and negative level generators. The level zero generators of $I_c(E_{11})$ are the first entry in equation (6.1), are those of $SO(11)$, which is consistent with the fact that the Cartan involution invariant subalgebra of $SL(D)$ is $SO(D)$. The Cartan involution invariant subalgebras for the real forms of the algebras in the Cartan-Killing list are the maximally compact algebras, for example, the Cartan involution invariant subalgebra of E_7 is $I_c(E_7) = SU(8)$ and for E_8 it is $I_c(E_8) = SO(16)$. A detailed discussion of the algebra $I_c(E_{11})$ and the more complete mathematical definition of the Cartan involution for any Kac-Moody algebra can be found in chapter sixteen of reference [16].

The classification, and many of the properties of Kac-Moody algebras, are usually investigated by taking the algebras to be over the complex numbers. However, in the application we have in mind we will use a particular real form. It is simpler to take the real form such that we find $SL(11)$ and then do a Wick rotation at the end of the calculation to find $SO(1,10)$ in $I_c(E_{11})$. However, one can also take a variant of the usual Cartan involution such that the Cartan involution invariant subalgebra $I_c(E_{11})$ contains $SO(1,10)$ [26] and the work with the Lorentz group from the very beginning. We largely, but not always, follow the former path.

The non-linear realisation of $E_{11} \otimes_s l_1$ with local subalgebra $I_c(E_{11})$ is constructed from a group element of $E_{11} \otimes_s l_1$ which can be written in the form $g = g_l g_E$ where [23, 24]

$$g_E = \dots e^{h_{a_1 \dots a_8, b} R^{a_1 \dots a_8, b}} e^{A_{a_1 \dots a_6} R^{a_1 \dots a_6}} e^{A_{a_1 \dots a_3} R^{a_1 \dots a_3}} e^{h_a{}^b K^a{}_b} \equiv e^{A_{\underline{\alpha}} R^{\underline{\alpha}}} \quad (6.2)$$

and

$$g_l = e^{x^a P_a} e^{x_{ab} Z^{ab}} e^{x_{a_1 \dots a_5} Z^{a_1 \dots a_5}} \dots \equiv e^{z^A L_A} \quad (6.3)$$

We have used the local subalgebra $I_c(E_{11})$ of equation (6.1) to choose the group element g_E to have no negative level generators. In doing this we used all of the local symmetry except for the local Lorentz transformation which remain a local symmetry. Apart from the level zero generators, the group element g_E lies in the Borel subalgebra of E_{11} .

The corresponding theory will contain the fields [23]

$$h_a{}^b, A_{a_1\dots a_3}, A_{a_1\dots a_6}, h_{a_1\dots a_8,b}, \dots \quad (6.4)$$

which depend on a spacetime that has the coordinates [24]

$$x^a, x_{ab}, x_{a_1\dots a_5}, x_{a_1\dots a_8}, x_{a_1\dots a_7,b}, \dots \quad (6.5)$$

The next few higher level coordinates can be read off from equation (5.6).

Thus one finds at level zero and one the fields of the usual formulation of eleven dimensional supergravity theory; the graviton and the three form. The field at level two is a six form which is well known to provide an equivalent description of the degrees of freedom that are usually carried by the three form. The field at level three $h_{a_1\dots a_8,b}$, provides a dual description of gravity; indeed it was in [23] that the linearised equation of motion, that was first order in spacetime derivatives and that express this duality were formulated for the field $h_{a_1\dots a_{D-3},b}$ in any dimension. These equations guaranteed that the dual fields really did describe gravity at the linearised level. However, above level three the non-linear realisation contains an infinite number of fields. The physical role of these higher level fields was unfamiliar to us at the early stages of work on E_{11} , but we now understand this role for quite large classes of the fields. We will discuss this later in the review but this still leaves many fields whose role is unknown.

The spacetime possess the usual coordinates of eleven dimensional spacetime, but it also has many more coordinates, in fact it is an infinite dimensional spacetime. An initially somewhat intimidating prospect. It can be shown that for every element in the Borel subgroup of E_{11} there is at least one corresponding element in the l_1 representation [25]. For example $K^a{}_b$ and $R^{a_1 a_2 a_3}$ correspond to P_a and $Z^{a_1 a_2}$ respectively; the general pattern being that one just knocks an index off the generators in the Borel subalgebra of E_{11} . However in the non-linear realisation every element in the Borel subalgebra of E_{11} leads to a field and every element in the l_1 representation leads to a coordinate of the spacetime. As a result we find that every field leads to at least one coordinate, a fact that is evident at low levels and is given in the correspondence below.

$$\begin{aligned} x^a &\leftrightarrow P_a \leftrightarrow K^a{}_b \leftrightarrow h_a{}^b \\ x_{a_1 a_2} &\leftrightarrow Z^{a_1 a_2} \leftrightarrow R^{a_1 a_2 a_3} \leftrightarrow A_{a_1 a_2 a_3} \\ x_{a_1 \dots a_5} &\leftrightarrow Z^{a_1 \dots a_5} \leftrightarrow R^{a_1 \dots a_6} \leftrightarrow A_{a_1 \dots a_6} \\ x_{a_1 \dots a_8}, x_{a_1 \dots a_7, b} &\leftrightarrow Z^{a_1 \dots a_8}, Z^{a_1 \dots a_7, b} \leftrightarrow R^{a_1 \dots a_8, b} \leftrightarrow h_{a_1 \dots a_8, b} \end{aligned}$$

We see from the above correspondence that the graviton is associated with the usual coordinates of spacetime x^a which carries the effects of gravity through the curvature of spacetime. The three form field is associated with the two form coordinates, the six form with the five form coordinate and so on. What this implies is that the E_{11} symmetry which

rotates the graviton into the three form and the higher fields also requires an extension of our notion of spacetime with a corresponding new geometry associated with the new fields beyond those of gravity. In this context it is interesting to recall the following quote taken from Salam's nobel lecture [30];

"...But are all the fundamental forces gauge forces? Can they be understood as such, in terms of charges- and their corresponding currents-only? And if they are how many charges? What unified entity are the charges components of? what is the nature of charge? Just as Einstein comprehended the nature of the gravitational charge in terms of space-time curvature, can we comprehend the nature of other charges-the nature of the entire unified set, as a set, in terms of something equally profound? This briefly is the dream..."

We now outline how to construct the dynamics of the $E_{11} \otimes_s l_1$ non-linear realisation. The Cartan forms were defined in section two in equations (2.13-15) for the case of a general type III non-linear realisation and in this case they can be written as

$$\mathcal{V} = dx^\Pi E_{\Pi}{}^A l_A + dx^\Pi G_{\Pi, \underline{\alpha}} R^{\underline{\alpha}} \quad (6.6)$$

where $G_{\Pi, \underline{\alpha}}$ are the Cartan forms of E_{11} . As previously noted we now denote the generators of E_{11} by $R^{\underline{\alpha}}$; the use of the underline being required to avoid ambiguity with the index α that arises in the discussion of the supergravities theories in lower dimensions (see next section).

The transformations of the Cartan forms under the symmetries of the non-linear realisation were given in equations (2.15) and (2.16). Although the Cartan forms, when viewed as forms, are inert under rigid transformations, the rigid transformations do act on the coordinate differentials, that is, on the dx^Π , contained in the Cartan form. This action induces a corresponding rigid E_{11} transformation on the lower index of $E_{\Pi}{}^A$. Indeed, it follows from equation (2.11) that the coordinates are inert under the local transformations but transform under the rigid transformations as

$$z^A l_A \rightarrow z^{A'} l_A = g_0 z^A l_A g_0^{-1} = z^\Pi D(g_0^{-1})_{\Pi}{}^A l_A \quad (6.7)$$

When written in matrix form the differential transformations act as $dz^T \rightarrow dz^{T'} = dz^T D(g_0^{-1})$. As a result the derivative $\partial_\Pi \equiv \frac{\partial}{\partial z^\Pi}$ in the generalised space-time transforms as $\partial'_\Pi = D(g_0)_{\Pi}{}^\Lambda \partial_\Lambda$.

A local $I_c(E_{11})$ transformation acts on the $\underline{\alpha}$ index of $G_{\Pi, \underline{\alpha}}$ and on the A index of $E_{\Pi}{}^A$ as governed by equation (2.16). As a result the rigid and local transformations of the object $E_{\Pi}{}^A$ can be summarise as

$$E_{\Pi}{}^{A'} = D(g_0)_{\Pi}{}^\Lambda E_{\Lambda}{}^B D(h)_B{}^A \quad (6.8)$$

and for its inverse by

$$(E^{-1})_A{}^{\Pi'} = D(h^{-1})_A{}^B (E^{-1})_B{}^\Lambda D(g_0^{-1})_\Lambda{}^\Pi \quad (6.9)$$

where $h^{-1} l_A h \equiv D(h)_A{}^B l_B$. Thus the object $E_{\Pi}{}^A$ transforms under a local $I_c(E_{11})$ transformation on its A index and by a rigid E_{11} induced coordinate transformation of the

generalised space-time on its Π index. These transformations mean that we can interpret E_Π^A as a vielbein of the space-time which possess the tangent group $I_c(E_{11})$. As we noted above, at level zero $I_c(E_{11})$ is just $SO(1,10)$. The reader may be puzzled by the use of the indices Π, Λ, \dots rather than A, B, \dots to label the elements of the l_1 representation, but this just reflects whether the indices transform under the rigid, or local transformations, that is, are world or tangent indices respectively.

Similarly, the object $G_{\Pi, \underline{\alpha}}$ transforms by the same a rigid E_{11} induced coordinate transformation on its Π index and by under a local $I_c(E_{11})$ transformation on its $\underline{\alpha}$ index. As a result we find that the object $G_{A, \underline{\alpha}} \equiv (E^{-1})_A^\Pi G_{\Pi, \underline{\alpha}}$ is inert under the rigid $E_{11} \otimes_s l_1$ transformations and only transforms under the $I_c(E_{11})$ transformations. To find the dynamics we can use the objects $G_{A, \underline{\alpha}}$, in this case the equations will be automatically invariant under the rigid transformations and we only need to solve the problem of finding a set of equations which is invariant under $I_c(E_{11})$ transformations.

It is very straightforward to compute the vielbein using the form of the group element of equation (6.2) and its definition as given in equation (6.6). One finds that the vielbein up to level two is given by [31,32]

$$E = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_\mu^a & -3e_\mu^c A_{cb_1 b_2} & 3e_\mu^c A_{cb_1 \dots b_5} + \frac{3}{2} e_\mu^c A_{[b_1 b_2 b_3} A_{|c| b_4 b_5]} \\ 0 & (e^{-1})_{[b_1}^{\mu_1} (e^{-1})_{b_2]}^{\mu_2} & -A_{[b_1 b_2 b_3} (e^{-1})_{b_4}^{\mu_1} (e^{-1})_{b_5]}^{\mu_2} \\ 0 & 0 & (e^{-1})_{[b_1}^{\mu_1} \dots (e^{-1})_{b_5]}^{\mu_5} \end{pmatrix} \quad (6.10)$$

Using the form of the group element of equation (6.2), the Cartan forms of equation (6.6) can readily be found to be given, up to level three, by [23,31]

$$G_a^b = (e^{-1} de)_a^b, \quad G_{a_1 \dots a_3} = e_{a_1}^{\mu_1} \dots e_{a_3}^{\mu_3} dA_{\mu_1 \dots \mu_3}, \quad (6.11)$$

$$G_{a_1 \dots a_6} = e_{a_1}^{\mu_1} \dots e_{a_6}^{\mu_6} (dA_{\mu_1 \dots \mu_6} - A_{[\mu_1 \dots \mu_3} dA_{\mu_4 \dots \mu_6]}) \quad (6.12)$$

where we are writing the quantities as forms.

In order to better understand some of the early E_{11} papers it is instructive to recall their progress towards constructing the non-linear realisation. Taking reference [22] and using [23] together one finds that the eleven dimensional supergravity was constructed, at very low levels, as a non-linear realisation of E_{11} ; it was shown to be a non-linear realisation of a particular algebra in reference [22] and the generators in this algebra are identified as those of E_{11} in reference [23]. However, this calculation suffered from a number of shortcomings. It only introduced the usual spacetime translation generators, which was not a E_{11} covariant procedure, and consequently the resulting field theory only possessed the usual spacetime. Also it only enforced the $I_c(E_{11})$ symmetry at the lowest level, that is, the very weak Lorentz part. As a result, the non-linear realisation carried out with these limitations did not lead uniquely to eleven dimensional supergravity and one had to fix several constants whose values were not determined by the calculation. A more systematic approach was taken in references [31,48, 33] and [34] where the non-linear realisation of $E_{11} \otimes_s l_1$ at low levels was constructed for the fields up to an including the dual graviton as well as the low level coordinates of the l_1 representation. These references enforced not only the Lorentz group symmetries of $I_c(E_{11})$ but also the much more powerful

symmetries at the next levels. The approach of reference [31] focused on finding duality equations which were first order in derivatives and it found the correct equations for the forms which were uniquely determined but there were unresolved issues with the graviton sector. In references [33, 34] the invariant second order equations were found, they were unique and when one retained only the low levels fields and the level zero coordinates they were precisely those of eleven dimensional supergravity. This essentially proved the E_{11} conjecture.

We recall that the Cartan involution invariant subalgebra $I_c(E_{11})$ at lowest level is $SO(1,10)$. At the next level $I_c(E_{11})$ possess a group element h which involves the generators at levels ± 1 and it is of the form [31]

$$h = 1 - \Lambda_{a_1 a_2 a_3} S^{a_1 a_2 a_3}, \quad \text{where} \quad S^{a_1 a_2 a_3} = R^{a_1 a_2 a_3} - \eta^{a_1 b_1} \eta^{a_2 b_2} \eta^{a_3 b_3} R_{b_1 b_2 b_3} \quad (6.13)$$

Under this transformation the Cartan forms of equation (6.6), when written as forms, change as

$$\delta \mathcal{V}_E = [S^{a_1 a_2 a_3} \Lambda_{a_1 a_2 a_3}, \mathcal{V}_E] - S^{a_1 a_2 a_3} d\Lambda_{a_1 a_2 a_3}. \quad (6.14)$$

The local $I_c(E_{11})$ variations of the Cartan forms are straightforward to compute, using the E_{11} algebra and they are given by [31,33,34]

$$\delta G_a{}^b = 18\Lambda^{c_1 c_2 b} G_{c_1 c_2 a} - 2\delta_a^b \Lambda^{c_1 c_2 c_3} G_{c_1 c_2 c_3}, \quad (6.15)$$

$$\delta G_{a_1 a_2 a_3} = -\frac{5!}{2} G_{b_1 b_2 b_3 a_1 a_2 a_3} \Lambda^{b_1 b_2 b_3} - 3G_{[a_1}^c \Lambda_{c|a_2 a_3]} - d\Lambda_{a_1 a_2 a_3} \quad (6.16)$$

$$\delta G_{a_1 \dots a_6} = 2\Lambda_{[a_1 a_2 a_3} G_{a_4 a_5 a_6]} - 8.7.2 G_{b_1 b_2 b_3 [a_1 \dots a_5, a_6]} \Lambda^{b_1 b_2 b_3} + 8.7.2 G_{b_1 b_2 [a_1 \dots a_5 a_6, b_3]} \Lambda^{b_1 b_2 b_3} \quad (6.17)$$

The above transformations do not take account of the fact that the l_1 index on the Cartan forms can transform. As explained above if this index is made into a tangent index, that is, $G_{A,\underline{a}} = (E^{-1})_A{}^\Pi G_{\Pi,\underline{a}}$ it transforms only under the local $I_c(E_{11})$ transformations, the transformation just being that for the inverse vielbein of equation (6.8). One finds that the Cartan forms, when referred to the tangent space, transforms on their l_1 index as [31,33,34]

$$\delta G_{a,\bullet} = -3G^{b_1 b_2}{}_{,\bullet} \Lambda_{b_1 b_2 a}, \quad \delta G^{a_1 a_2}{}_{,\bullet} = 6\Lambda^{a_1 a_2 b} G_{b,\bullet}, \dots \quad (6.18)$$

Of course to get the full transformation one must combine the transformations of equation (6.18) with those of equations (6.12-6.17); for example we find that

$$\delta G^{e_1 e_2}{}_{,a}{}^b = 18\Lambda^{c_1 c_2 b} G^{e_1 e_2}{}_{,c_1 c_2 a} - 2\delta_a^b \Lambda^{c_1 c_2 c_3} G^{e_1 e_2}{}_{,c_1 c_2 c_3} + 6\Lambda^{e_1 e_2 d} G_{d,a}{}^b \quad (6.19)$$

The detailed construction of the equations of motion which follow from the $E_{11} \otimes_s l_1$ non-linear realisation was given in reference [34] following earlier results in references [33] and [31]. We refer the reader to this reference and confine ourselves here to stating the result. One finds the unique equations of motion are given by

$$\mathcal{E}^{a_1 a_2 a_3} \equiv \frac{1}{2} G_{b,d}{}^d G^{[b,a_1 a_2 a_3]} - 3G_{b,d}{}^{[a_1} G^{b,d|a_2 a_3]} - G_{c,b}{}^c G^{[b,a_1 a_2 a_3]} + (\det e)^{\frac{1}{2}} e_b{}^\mu \partial_\mu G^{[b,a_1 a_2 a_3]}$$

$$\begin{aligned}
& + \frac{1}{2 \cdot 4!} \epsilon^{a_1 a_2 a_3 b_1 \dots b_8} G_{[b_1, b_2 b_3 b_4]} G_{[b_5, b_6 b_7 b_8]} \\
& - 9 G^{ca_1}_{, cd_1 d_2} G^{[d_1, d_2 a_2 a_3]} + \frac{5}{16} \epsilon^{a_1 a_2 a_3 b_1 \dots b_8} G_{b_1, b_2 b_3 b_4} G^{c_1 c_2}_{, c_1 c_2 b_5 \dots b_8} \\
& + \frac{1}{4} e_{\mu_1}^{[a_1} e_{\mu_2}^{a_2} e_{\mu_3}^{a_3]} \partial_\nu \left((\det e)^{\frac{1}{2}} G^{\mu_1 \mu_2, \nu \mu_3} \right) + \frac{1}{4} (\det e)^{\frac{1}{2}} \omega_{\nu, [a_1 | b} G^{a_2 a_3], b \nu} \\
& + \frac{1}{4} G^{[a_1 a_2]_{, d}^d} (G^{[a_3]_{, c}^c} - G_{c, [a_3] c}) - \frac{1}{4} \partial_\nu \left((\det e)^{\frac{1}{2}} (G^{[a_1 a_2]_{, d}^d} e^{\nu [a_3]} - G^{[a_1 a_2]_{, [a_3] \nu}}) \right) \\
& + \frac{1}{2} (G^{[a_1, c}{}^{a_2} G^{c | a_3]_{, d}^d} - G^{c [a_1, a_2 | e] G_{e, c}{}^{a_3]}) \\
& + \frac{15}{2} e_{\mu_1}^{[a_1} e_{\mu_2}^{a_2} e_{\mu_3}^{a_3]} \partial_\nu \left((\det e)^{\frac{1}{2}} G^{d_1 d_2}_{, d_1 d_2}{}^{\nu \mu_1 \mu_2 \mu_3} \right) \\
& + e_{\mu_1}^{[a_1} e_{\mu_2}^{a_2} e_{\mu_3}^{a_3]} \left(\frac{1}{2} (\det e)^{\frac{1}{2}} g_{\tau \sigma} g^{\mu_1 \lambda} \partial_\lambda G^{\tau \mu_2, (\sigma \mu_3)} - \frac{1}{2} G^{\tau \mu_1}_{, d}{}^d G^{\mu_2, (\mu_3 \tau)} \right. \\
& \left. - \frac{1}{4} G^{\tau \mu_1, (\mu_2 \tau)} G^{\mu_3}_{, d}{}^d - G^{\tau \mu_1}_{, (\tau \sigma)} G^{\mu_2, (\mu_3 \sigma)} + G^{\tau \mu_1, (\mu_2 \sigma)} G_{\sigma, (\tau \mu_3)} \right) = 0 \quad (6.20)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{ab} & \equiv (\det e) \mathcal{R}_{ab} - 12 \cdot 4 G_{[a, c_1 c_2 c_3]} G^{[e, c_1 c_2 c_3]} \eta_{eb} + 4 \eta_{ab} G_{[c_1, c_2 c_3 c_4]} G^{[c_1, c_2 c_3 c_4]} \\
& - 3 \cdot 5! G^{d_1 d_2}_{, d_1 d_2 a}{}^{c_1 c_2 c_3} G_{[b, c_1 c_2 c_3]} - 3 \cdot 5! G^{d_1 d_2}_{, d_1 d_2 b}{}^{c_1 c_2 c_3} G_{[a, c_1 c_2 c_3]} \\
& + \frac{5!}{2} \eta_{ab} G^{d_1 d_2}_{, d_1 d_2 c_1 \dots c_4} G^{[c_1, c_2 c_3 c_4]} - 12 G^{c_1 c_2}_{, a}{}^{c_3} G_{[b, c_1 c_2 c_3]} + 3 G^{c_1 c_2}_{, e}{}^e G_{[a, b c_1 c_2]} \\
& - 6 (\det e) e_b{}^\mu e_a{}^\lambda \partial_{[\mu} \left[(\det e)^{-\frac{1}{2}} G^{\tau_1 \tau_2}_{, |\lambda \tau_1 \tau_2|} \right] \\
& - (\det e)^{\frac{1}{2}} \omega_{c, b}{}^c G^{d_1 d_2}_{, d_1 d_2 a} - 3 (\det e)^{\frac{1}{2}} \omega_{a, b}{}^c G^{d_1 d_2}_{, d_1 d_2 c} = 0 \quad (6.21)
\end{aligned}$$

where

$$\mathcal{R}_a{}^b = e_a{}^\mu \partial_\mu \Omega_\nu{}^{bd} e_d{}^\nu - e_a{}^\mu \partial_\nu \Omega_\mu{}^{bd} e_d{}^\nu + \Omega_{a,}{}^b{}_c \Omega_d{}^{cd} - \Omega_{d,}{}^b{}_c \Omega_a{}^{cd}, \quad (6.22)$$

$$(\det e)^{\frac{1}{2}} \omega_{c, ab} = -G_{a, (bc)} + G_{b, (ac)} + G_{c, [ab]} \quad (6.23)$$

and

$$(\det e)^{\frac{1}{2}} \Omega_{c, ab} = (\det e)^{\frac{1}{2}} \omega_{c, ab} - 3 G^{dc}_{, dab} - 3 G^d_{b, dac} + 3 G^d_{a, dbc} - \eta_{bc} G^{d_1 d_2}_{, d_1 d_2 a} + \eta_{ac} G^{d_1 d_2}_{, d_1 d_2 b} \quad (6.24)$$

We have corrected the sign of the eleventh term compared to that in reference [34].

Under the $I_c(E_{11})$ transformation of equation (6.13) they transform as

$$\delta \mathcal{E}^{a_1 a_2 a_3} = \frac{3}{2} E_b{}^{[a_1 | \Lambda^{b | a_2 a_3]}$$

$$\begin{aligned}
& + \frac{1}{24} e_{\mu_1}^{a_1} e_{\mu_2}^{a_2} e_{\mu_3}^{a_3} \epsilon^{\mu_1 \mu_2 \mu_3 \nu \lambda_1 \dots \lambda_4 \tau_1 \tau_2 \tau_3} \partial_\nu \left((\det e)^{-\frac{1}{2}} E_{\lambda_1 \dots \lambda_4} g_{\tau_1 \kappa_1} g_{\tau_2 \kappa_2} g_{\tau_3 \kappa_3} \right) \Lambda^{\kappa_1 \kappa_2 \kappa_3} \\
& + \frac{1}{24 \cdot 4!} \epsilon^{a_1 a_2 a_3 b_1 \dots b_8} \epsilon_{b_1 \dots b_4 c_1 c_2 c_3 e_1 \dots e_4} E_{b_5 \dots b_8} G^{[e_1, e_2 \dots e_4]} \Lambda^{c_1 c_2 c_3}
\end{aligned} \tag{6.25}$$

and

$$\begin{aligned}
\delta \mathcal{E}_{ab} = & -36 \Lambda^{d_1 d_2} {}_a E_{b d_1 d_2} - 36 \Lambda^{d_1 d_2} {}_b E_{a d_1 d_2} + 8 \eta_{ab} \Lambda^{d_1 d_2 d_3} E_{d_1 d_2 d_3} \\
& - 2 \epsilon_a{}^{c_1 c_2 c_3 e_1 \dots e_4} f_1 f_2 f_3 \Lambda_{f_1 f_2 f_3} E_{e_1 \dots e_4} G_{[b, c_1 c_2 c_3]} - 2 \epsilon_b{}^{c_1 c_2 c_3 e_1 \dots e_4} f_1 f_2 f_3 \Lambda_{f_1 f_2 f_3} E_{e_1 \dots e_4} G_{[a, c_1 c_2 c_3]} \\
& + \frac{1}{3} \eta_{ab} \epsilon^{c_1 \dots c_4 e_1 \dots e_4} f_1 f_2 f_3 E_{e_1 \dots e_4} G_{[c_1, c_2 c_3 c_4]} \Lambda_{f_1 f_2 f_3}
\end{aligned} \tag{6.26}$$

In these variations

$$E_{a_1 \dots a_4} \equiv \mathcal{G}_{[a_1, a_2 a_3 a_4]} - \frac{1}{2 \cdot 4!} \epsilon_{a_1 a_2 a_3 a_4}{}^{b_1 \dots b_7} G_{b_1, b_2 \dots b_7} = 0 \tag{6.27}$$

where

$$\mathcal{G}_{a_1, a_2 a_3 a_4} \equiv G_{[a_1, a_2 a_3 a_4]} + \frac{15}{2} G^{b_1 b_2}{}_{, b_1 b_2 a_1 \dots a_4} \tag{6.28}$$

This is the first order duality relation between the three form and six form fields. It can be found independently by requiring an equation which is first order in derivatives and contains the three form and is part of an invariant set of equations under the transformations of (6.13) [31,34]. When carrying out the variation of this equation one also finds the duality relation between the usual graviton and the dual graviton. We note that taking the spacetime derivative we find, at least, to lowest orders the second order equation of motion of equation (6.20).

If we discard the derivatives with respect to the higher level coordinates we find that the above equations of motion can be written as

$$\partial_\nu ((\det e)^{\frac{1}{2}} G^{[\nu, \mu_1 \mu_2 \mu_3]}) + \frac{1}{2 \cdot 4!} (\det e)^{-1} \epsilon^{\mu_1 \mu_2 \mu_3 \tau_1 \dots \tau_8} G_{[\tau_1, \tau_2 \tau_3 \tau_4]} G_{[\tau_5, \tau_6 \tau_7 \tau_8]} = 0 \tag{6.29}$$

and that

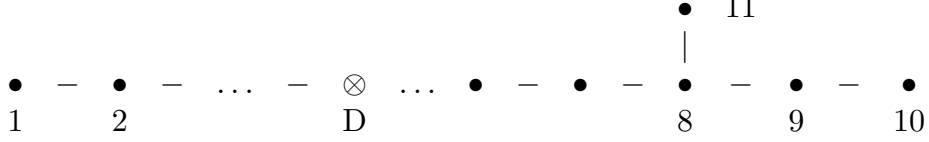
$$E_a{}^b \equiv (\det e) R_a{}^b - 12 \cdot 4 G_{[a, c_1 c_2 c_3]} G^{[b, c_1 c_2 c_3]} + 4 \delta_a^b G_{[c_1, c_2 c_3 c_4]} G^{[c_1, c_2 c_3 c_4]} = 0 \tag{6.30}$$

We recognise these are the equations of motion of eleven dimensional supergravity. Thus the $E_{11} \otimes_s l_1$ non-linear realisation lead to unique equations, at least up to the levels studied, and when truncated to contain the low level fields and only the usual coordinates of spacetime these equations of motion are precisely those of the bosonic sector of eleven dimensional supergravity. The reader who repeats even parts of these calculations will be left in no doubt as the validity of the E_{11} approach.

7 The $E_{11} \otimes_s l_1$ non-linear realisation in D dimensions

In the section five we considered the $E_{11} \otimes_s l_1$ algebra when decomposed with respect to its $GL(11)$ subalgebra and we found that the $E_{11} \otimes_s l_1$ non-linear realisation, when decomposed in this way, was an eleven dimensional theory that at low levels was precisely

eleven dimensional supergravity. To find the theory in D dimensions we delete the node labelled D in the E_{11} Dynkin diagram to find the residual algebra $GL(D) \otimes E_{11-D}$, we then decompose the $E_{11} \otimes_s l_1$ algebra into representations of this subalgebra and then construct the corresponding non-linear realisation [35,36,37,38].



In this non-linear realisation the $GL(D)$ subalgebra will lead to gravity in D dimensions, confirming the fact that the resulting theory is indeed in D dimensions. The E_{11-D} subalgebra is the well known U duality algebra of the supergravity theory in D dimensions.

Carrying out the decomposition we find at low levels exactly the fields of the D dimensional maximal supergravity theory and a generalised spacetime whose level zero part is just the usual spacetime in D dimensions. For example, in five dimensions one deletes node five to find the remaining algebra $GL(5) \otimes E_6$ and decomposing with respect to this subalgebra one finds the resulting non-linear realisation has the field content [38,33]

$$h_a^b, \varphi_\alpha, A_{aM}, A_{a_1 a_2}^N, A_{a_1 a_2 a_3, \alpha}, A_{a_1 a_2, b}, \dots \quad (7.1)$$

and the spacetime has the coordinates [38,33]

$$x^a, x_N, x_a^N, x_{a_1 a_2, \alpha}, x_{ab}, \dots \quad (7.2)$$

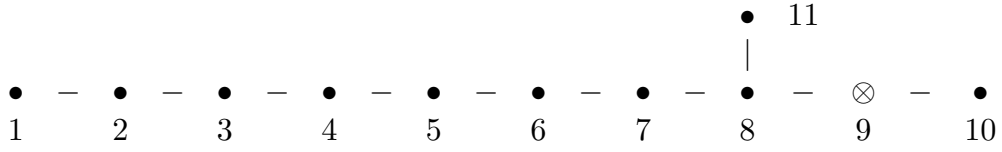
For all these objects the lower (upper) case indexes $a, b, c, \dots = 1, \dots, 5$ correspond to 5 ($\bar{5}$)-dimensional fundamental representation of $GL(5)$. The indexes $\alpha, \beta, \gamma, \dots = 1, \dots, 78$ correspond to 78-dimensional adjoint representation of E_6 and the upper and lower case indexes $N, M, P, \dots = 1, \dots, 27$ correspond to $\bar{27}$ -dimensional and 27-dimensional representations of E_6 respectively.

As in eleven dimensions the fields and coordinates are classified by a level. However the definition of the level depends on the node being deleted, we refer the reader to reference [16] for a detailed account. For theories in less than ten dimensions the level of the fields is just the number of lower minus upper $GL(D)$ indices. While for the coordinates it is the same but minus one. The fields in five dimensions, given in equation (7.1), are the graviton and the scalars at level zero, while at level one we find the vectors. Thus we find the bosonic fields of the usual description of five-dimensional supergravity. The level two fields provide a dual description of the vectors and the two fields at level three are a dual description of the scalars and the graviton respectively. The equations of motion that follow from the $E_{11} \otimes_s l_1$ non-linear realisation leading to the five dimensional theory were found, at low levels, in reference [33]. They were the equations of motion of five dimensional maximal supergravity. In reference [33] some undetermined constants appear but they are fixed to the required values by considering the dimensional reduction of the unique eleven dimensional equations of motion which follow from the $E_{11} \otimes_s l_1$ non-linear realisation [34].

The reader can find an account of the $E_{11} \otimes_s l_1$ non-linear realisation in the decomposition that leads to four dimensions in reference [48] where the equations of motion for the form fields are derived and are found to agree with those of maximal supergravity in four dimensions. The equations of motion of the gravity sector is only partially computed but the way is not clear to apply the techniques of reference [33] and [34] and find the gravity equations. It is inevitable that it will agree with the equation of motion of the maximal supergravity theory in four dimensions.

An exception to the above discussion is provided by ten dimensions. To find such a theory one has to find ten dimensional gravity and so a $GL(10)$ subalgebra, which includes an A_9 subalgebra whose Dynkin diagram consists of nine dots in a row. Looking at the E_{11} Dynkin diagram and starting from node one it is apparent that, unlike in less than ten dimensions where there is only one possibility, there are two possibilities.

The first possibility is to delete node nine [35]



This leads to the algebra $GL(10) \otimes SL(2)$. The $SL(10)$ of the $GL(10)$ arises from the dots one to eight as well as dot eleven. In general we refer to the line of dots that is associated with gravity as the gravity line. One finds that the field content of the resulting non-linear realisation is given by [35, 39]

$$h_a{}^b, \phi, \chi; A_{a_1 a_2}^\alpha; A_{a_1 \dots a_4}; A_{a_1 \dots a_6}^\alpha; A_{a_1 \dots a_8}^{(\alpha\beta)}, h_{a_1 \dots a_7, b}; A_{a_1 \dots a_{10}}^{(\alpha\beta\gamma)}, A_{a_1 \dots a_8, b_1 b_2}^\alpha, A_{a_1 \dots a_9, b}^\alpha; \dots \quad (7.3)$$

where $a, b, \dots = 0, 1, \dots, 9$ are $GL(10)$ indices and $\alpha, \beta = 1, 2$ are $SL(2)$ indices. The first listed fields, the graviton, the scalars, the doublet two forms and the four form are those of the usual description of IIB supergravity. Node ten in the above Dynkin diagram is not connected to the gravity line and so leads to an $SL(2)$ symmetry which is just the $SL(2)$ symmetry of the IIB theory. The $A_{a_1 \dots a_6}^\alpha$ fields are the duals of the two forms. The triplet of eight forms are the duals of the scalar fields and had previously been discussed in the context of the IIB theory [41] but the $\bar{4}$ representation of $SL(2)$ ten forms were a prediction of E_{11} [39, 40] and their presence was confirmed from the supersymmetry perspective in [42]. The field $h_{a_1 \dots a_7, b}$ is the dual graviton. Higher level fields can be found in Table 17.5.3 on page 596 of reference [16].

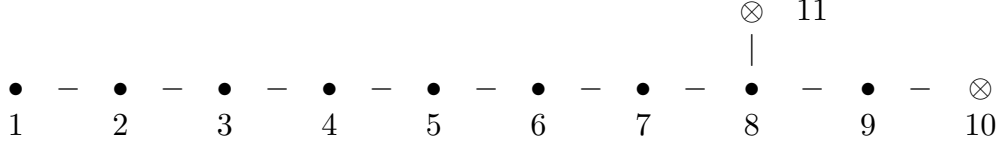
The non-linear realisation leads to a spacetime with the coordinates

$$\begin{aligned} x^a; x_a{}^\alpha; x_{a_1 a_2 a_3}; x_{a_1 \dots a_5}^\alpha; x_{a_1 \dots a_7}, x_{a_1 \dots a_7}^{(\alpha\beta)}, x_{a_1 \dots a_6, b}; x_{a_1 \dots a_9}^\alpha(2), x_{a_1 \dots a_9}^{(\alpha\beta\gamma)}, \\ x_{a_1 \dots a_8, b}^\alpha(2), x_{a_1 \dots a_7, b_1 b_2}^\alpha; x_a(3), x_a^{(\alpha\beta)}(3), x_{a_1 \dots a_9, b_1 b_2}, x_{a_1 \dots a_9, b_1 b_2}^{(\alpha\beta)}, x_{a_1 \dots a_9, (b_1 b_2)}, \\ x_{a_1 \dots a_8, b_1 b_2 b_3}, x_{a_1 \dots a_8, b_1 b_2 b_3}^{\alpha\beta}, x_{a_1 \dots a_8, b_1 b_2, c}, x_{a_1 \dots a_7, b_1 \dots b_4} \end{aligned} \quad (7.4)$$

where the number in brackets give the multiplicities and if there is no bracket the multiplicity is one. All the coordinates belong to irreducible representations of $SL(10)$. The

level is the number of down minus up GL(10) indices divided by two for the fields and the same for the coordinates except that one must subtract one first.

For the second possibility, we delete node ten to find a SO(10,10) subalgebra and then delete node eleven which leads to the required GL(10) subalgebra [23]. The gravity line is made up of nodes one to nine.



The corresponding field content in the non-linear realisation is given by [39]

$$\begin{aligned}
& h_a{}^b(0), \phi(0), A_{a_1 a_2}(0); A_a, A_{a_1 a_2 a_3}(1), A_{a_1 \dots a_5}(1), A_{a_1 \dots a_7}(1), A_{a_1 \dots a_9}(1); \\
& A_{a_1 \dots a_6}(2), A_{a_1 \dots a_8}(2), A_{a_1 \dots a_{10}}(2), A_{a_1 \dots a_{10}}(2); h_{a_1 \dots a_7, b}(2), A_{a_1 \dots a_8, b_1 b_2}(2), A_{a_1 \dots a_8, b_1 b_2}(2), \\
& A_{a_1 \dots a_9, b}(2), A_{a_1 \dots a_9, b_1 b_2 b_3}(2); A_{a_1 \dots a_{10}, b_1 b_2}(2), A_{a_1 \dots a_{10}, b_1 \dots b_4}(2); \dots \quad (7.5)
\end{aligned}$$

where $a, b, \dots = 0, 1, \dots, 9$ and the number in the brackets denotes the level with respect to node ten. As a result those fields with the same level group into representations of SO(10,10). The fact that some fields are repeated indicates that they occur with the corresponding multiplicity.

The level zero fields of equation (7.5) are the graviton, the scalar and the two form which are just those of the massless NS-NS sector of the IIA superstring. The level one fields belong to the spinor representations of SO(10,10) and are the vector, the three form, five form, seven form and a nine form. The first two of these fields are those of the massless R-R sector of the IIA superstring while the five form and seven forms are the duals of the three and one forms respectively. As we will discuss later the nine form is associated with Romans theory. The dual graviton can be found at level two. Thus we find among these fields of the usual description of IIA supergravity their duals.

The spacetime, which is encoded in the non-linear realisation and that arises from this decomposition, has the coordinates

$$\begin{aligned}
& x^a, y_a, x, x_{a_1 a_2}, x_{a_1 \dots a_4}; x_{a_1 \dots a_5}, x_{a_1 \dots a_6}, x_{a_1 \dots a_6, b}, x_{a_1 \dots a_7}(2), x_{a_1 \dots a_8}(2), \\
& x_{a_1 \dots a_8, b}(2), x_{a_1 \dots a_9}(3), x_{a_1 \dots a_{10}}(4), x_{a_1 \dots a_7, b_1 b_2}, x_{a_1 \dots a_7, b}, x_{a_1 \dots a_9, b}(4), x_{a_1 \dots a_{10}, b}(3), \\
& x_{a_1 \dots a_9, b_1 b_2}(2), x_{a_1 \dots a_8, b_1 b_2 b_3}(2), x_{a_1 \dots a_8, b_1 b_2}(2), x_{a_1 \dots a_8, (b_1 b_2)}, x_{a_1 \dots a_7, b_1 b_2 b_3}, \\
& x_{a_1 \dots a_{10}, b_1 b_2}(7), x_{a_1 \dots a_{10}, (b_1 b_2)}(3), x_{a_1 \dots a_9, b_1 b_2 b_3}(5), x_{a_1 \dots a_9, b_1 b_2, c}(2), x_{a_1 \dots a_8, b_1 \dots b_4}(2), \\
& x_{a_1 \dots a_8, b_1 b_2 b_3, c}, x_{a_1 \dots a_7, b_1 \dots b_5}, x_{a_1 \dots a_{10}, b_1 b_2 b_3}(7), x_{a_1 \dots a_{10}, b_1 b_2, c}(4), x_{a_1 \dots a_{10}, (b_1 b_2 b_3)}, \\
& x_{a_1 \dots a_9, b_1 \dots b_4}(4), x_{a_1 \dots a_9, b_1 b_2 b_3, c}(3), x_{a_1 \dots a_8, b_1 \dots b_5}(2), x_{a_1 \dots a_7, b_1 \dots b_6}, x_{a_1 \dots a_8, b_1 \dots b_4, c}, \dots \quad (7.6)
\end{aligned}$$

where the number in brackets give the multiplicities and if there is no bracket the multiplicity is one. All the coordinates belong to irreducible representations of SL(10). The first two coordinates occur at level zero and belong to the vector representation of SO(10,10).

At level zero the $E_{11} \otimes_s l_1$ non-linear realisation when further decomposed into representations of $GL(10)$, as discussed above, contain the massless fields in the NS-NS sector of the superstring which live on a twenty dimensional spacetime with coordinates x^a and y_a of equation (7.6) and the detailed equations of motion were worked out in reference [43]. The result is the same as Siegel theory [44,45]. The more recent work on doubled field theory was shown to be equivalent to Siegel theory in reference [46]. The non-linear realisation up to and including level one contains the above fields but also the massless fields in the R-R sector of the superstring [47]. Indeed, it was in this paper that Siegel theory was extended to include the massless R-R fields and so all the massless fields of IIA supergravity.

8 $E_{11} \otimes_s l_1$ non-linear realisation as a unified theory

In this section we will explain that the $E_{11} \otimes_s l_1$ non-linear realisation contains all we know about maximal supergravities and so is a unified theory. In other words the many very different maximal supergravity theories are packaged up into this one theory.

The low level fields in the $E_{11} \otimes_s l_1$ non-linear realisation were listed above, however, it contains an infinite number of fields whose character is largely unknown. Nonetheless in this section we will list some of the higher level fields and find out what their role in the theory is. In particular, it is straightforward to find all the form fields, that is, fields whose indices are totally antisymmetrised, in D dimensions. These fields are listed in the table below [37,49]

Table 1. The forms fields in D dimension with their E_{11-D} representations.

D	E_{11-D}	A_a	$A_{a_1 a_2}$	$A_{a_1 a_2 a_3}$	$A_{a_1 \dots a_4}$	$A_{a_1 \dots a_5}$	$A_{a_1 \dots a_6}$	$A_{a_1 \dots a_7}$	$A_{a_1 \dots a_8}$
8	$SL(3) \otimes SL(2)$	$(\mathbf{3}, \mathbf{2})$	$(\mathbf{3}, \mathbf{1})$	$(\mathbf{1}, \mathbf{2})$	$(\mathbf{3}, \mathbf{1})$	$(\mathbf{3}, \mathbf{2})$	$(\mathbf{1}, \mathbf{3})$ $(\mathbf{8}, \mathbf{1})$	$(\mathbf{3}, \mathbf{2})$ $(\mathbf{6}, \mathbf{2})$	$(\mathbf{3}, \mathbf{1})$ $(\mathbf{15}, \mathbf{1})$ $(\mathbf{3}, \mathbf{1})$ $(\mathbf{3}, \mathbf{3})$
7	$SL(5)$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{40}}$ $\overline{\mathbf{15}}$	$\mathbf{70}$ $\mathbf{45}$ $\mathbf{5}$	- - -
6	$SO(5, 5)$	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$	$\mathbf{144}$	$\mathbf{320}$ $\overline{\mathbf{126}}$ $\mathbf{10}$	- - -	- - -
5	E_6	$\mathbf{27}$	$\overline{\mathbf{27}}$	$\mathbf{78}$	$\mathbf{351}$	$\overline{\mathbf{1728}}$ $\overline{\mathbf{27}}$	- -	- -	- -
4	E_7	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{8645}$ $\mathbf{133}$	- -	- -	- -	- -
3	E_8	$\mathbf{248}$	$\mathbf{3875}$ $\mathbf{1}$	$\mathbf{147250}$ $\mathbf{3875}$ $\mathbf{248}$	- - -	- - -	- - -	- - -	- - -

Looking at table 1 one sees that for every form field $A_{a_1 \dots a_n}$ of rank n with $n < \frac{D}{2}$ indices there is a dual field $A_{a_1 \dots a_{D-n-2}}$ that belongs to a conjugate representation of

the E_{11-D} algebra. One finds that the dynamics of the non-linear realisation leads to an equation that relates the fields strengths of these two fields through a duality relation which is first order in derivatives [50]. See reference [16] for a review of this point. This same pattern occurred in eleven dimensions and in the ten dimensional IIA and IIB theories discussed earlier.

Thus the non-linear realisation leads to a democratic formulation in that the different possible field descriptions of the degrees of freedom of the theory are present. For example, in eleven dimensions the degrees of freedom usually encoded by the three form A_3 can equally well be realised by the fields A_6 . It has been shown in eleven dimensions that at higher levels one finds in the $E_{11} \otimes_s l_1$ non-linear realisation the fields $A_{3,9}, A_{3,9,9}, A_{3,9,9,9}, \dots$ or $A_{6,9}, A_{6,9,9}, A_{6,9,9,9}, \dots$ where the numbers refer to the blocks of antisymmetric indices [51]. If we were to restrict the index range to be only over nine values, as one might suppose is the case in a light-cone analysis, then they all belong to the same representation of $SO(9)$ and so one might think that these fields also describe the same degrees of freedom as the original A_3 field. Thus the $E_{11} \otimes_s l_1$ non-linear realisation provides an infinite number of ways of describing the degrees of freedom of eleven dimensional supergravity. Indeed these different possible descriptions of the particles in the theory are rotated into each other under the E_{11} symmetry and so part of the E_{11} symmetry can be viewed as a vast duality symmetry. Similar conclusions apply in lower dimensions.

Table 1 also contains next to top forms which are forms that have $D - 1$ totally antisymmetrised $SL(D)$ indices. We note that these fields also carry particular representations of E_{11-D} . Suppressing these latter indices such a field in D dimensions has the form $A_{a_1 \dots a_{D-1}}$; the corresponding field strength is of the form $F_{a_1 \dots a_D}$ and it should appear in the action in the generic form

$$\int d^D x (\det e_\mu^a) F_{a_1 \dots a_D} F^{a_1 \dots a_D} \quad (7.7)$$

Its equation of motion is of the form $\partial_{\mu_1} ((\det e_\nu^a) F^{\mu_1 \dots \mu_D}) = 0$ and it has the solution $F_{a_1 \dots a_D} = m \epsilon_{a_1 \dots a_D}$ where m is a constant. Substituting this back into the action we find a cosmological constant. Thus the next to top forms lead to theories with cosmological constants and so the $E_{11} \otimes_s l_1$ non-linear realisation automatically contains theories with a cosmological constant which are classified by the representations of E_{11-D} to which the next to top forms belong.

Supergravity theories with a cosmological constant have been studied since the discovery of the first supergravity theory. To find them one essentially takes a known supergravity theory, adds by hand a cosmological constant and then tries to restore the supersymmetry by adding terms to the transformations rules and the action. It turns out that this is not possible for the eleven dimensional supergravity theory and the ten dimensional IIB theory, however, for the ten dimensional IIA theory there is a unique possibility called Romans theory [52]. For the lower dimensional maximal supergravity theories there are in fact many ways to proceed and so there are many different theories with a cosmological constant that preserve all the supersymmetries. These different theories gauge different parts of the E_{11-D} symmetry and as a result such theories have become known as gauged

supergravities. While some gauged supergravities can be obtained from ten or eleven dimensional supergravities by dimensional reduction on various manifolds, such as spheres, many have no known higher dimensional origin when viewed from the viewpoint of conventional supergravity. As a result they are not part of what is normally considered as M theory, since as we have explained, M theory is not a theory but a set of relations between theories and for these latter theories there is no connection to the theories that are usually considered part of M theory.

There are no such next to top forms in eleven dimensions and in the IIB theory which is consistent with the fact that these theories do not have an extension to include a cosmological constant. However, the $E_{11} \otimes_s l_1$ non-linear realisation when decomposed in a way that leads to the IIA theory in ten dimensions possess a nine form, see equation (7.5), which leads to a deformation of the IIA theory which possess a cosmological constant [39]. This is of course Romans theory.

Examining the next to top forms in table 1 for the theories in lower dimensions we find that in four dimensions they belong to the representation of dimension 912 of E_7 . Following our discussion just above we conclude that these fields lead to theories with cosmological constants which are classified by the 912 representations of E_7 . In general the representations of the next to top forms in D dimensions will classify all the possible gauged maximal supergravities in that dimension [37,49]. The result is in agreement with previous work carried out over many years and based on supersymmetry [53]. Indeed this latter work used the so called hierarchy method which introduces some of the form fields found in table 1. We note that although the next to top fields do not lead to new degrees of freedom, they clearly do lead to physical effects.

Table 1 also contains top forms, that is, forms with D totally antisymmetrised indices. These will not lead to dynamical degrees of freedom but they may well lead to physical effects. We note that they will occur as the lead term in the Wess-Zumino terms in brane actions.

As we have seen the different maximal supergravity theories arise from taking different decomposition of the $E_{11} \otimes_s l_1$ algebra and that within a given decomposition we can also find all the gauged supergravities by taking different next to top forms to be non-zero. However, there is only one E_{11} algebra and only one l_1 representation as such any two theories found by taking different decompositions are related to each other, that is, the fields in the different theories are related in a one to one manner and so are the coordinates [36]. It is straight forward to find the correspondence. In a given theory, or decomposition, every field component arises in the non-linear realisation from a given E_{11} generator and so from a given E_{11} root. To find the corresponding field component in any other theory one just has to find the one that corresponds to the same root. We note that the usual formulations of supergravity contain fields that appear at the lowest levels in the non-linear realisation and one can find that, even if the field in one theory is one of those that appears in the usual supergravity theory, the corresponding field in the other theory is one that is at higher level and does not appear in the usual description of this other supergravity theory. As similar argument applies to the coordinates in the different theories.

The correspondence between the different theories is especially interesting to examine for the gauged supergravity and in particular how a non-zero next to top form, which is

therefore responsible for the cosmological constant, is mapped into another field in a different theory. Indeed one can find what field in the eleven dimensional theory corresponds to a given next to top form in lower dimensions and as a result find the eleven dimensional origin of all gauged supergravities. We now give two examples of this procedure. Let us first consider Romans theory. As we discussed above this theory arises from the nine form of equation (7.5). However, rather than tracing to what field the E_{11} root of this field corresponds to in eleven dimensions it is simpler, in this case, to carry out the dimensional reduction from eleven dimensions directly. The $E_{11} \otimes_s l_1$ non-linear realisation when decomposed into representations of $GL(11)$ leads to the eleven dimensional fields [23, 55]

$$\begin{aligned} h_{\hat{a}}^{\hat{b}}(0), A_{\hat{a}_1 \hat{a}_2 \hat{a}_3}(1), A_{\hat{a}_1 \dots \hat{a}_6}(2), h_{\hat{a}_1 \dots \hat{a}_8, \hat{b}}(3), \\ A_{\hat{a}_1 \dots \hat{a}_9, \hat{b}_1 \hat{b}_2 \hat{b}_3}(4), A_{\hat{a}_1 \dots \hat{a}_{11}, \hat{b}}(4), A_{\hat{a}_1 \dots \hat{a}_{10}, (\hat{b}_1 \hat{b}_2)}(4), \dots \end{aligned} \quad (7.8)$$

where $\hat{a}, \hat{b}, \dots = 1, \dots, 11$ and the number in the brackets is the level. Carrying out the dimensional reduction to ten dimensions by hand we get the IIA fields and it is easy to see that the nine form in ten dimensions arises from the level four field $A_{a_1 \dots a_{10}, bc}$ which is antisymmetric in its a_1, \dots, a_{10} indices but symmetric in the b, c indices. Indeed the nine form arises as $A_{a_1 \dots a_9 11, 1111}$. We note that this level four field is in a part of the $E_{11} \otimes_s l_1$ non-linear realisation that is beyond the eleven dimensional supergravity theory. Thus we have found the eleven dimensional origin of Romans theory, something that could not have been found using conventional supergravity techniques.

Our second example concerns four dimensions where the next to top fields have the form $A_{a_1 a_2 a_3 \bullet}$ where \bullet refers to the 912- dimensional representation of E_7 to which this field belongs. Decomposing this representation to representations of $SL(8)$ we find that [37]

$$912 \rightarrow 420 \oplus \overline{420} \oplus 36 \oplus \overline{36} \quad (7.9)$$

which correspond to the tensors

$$\phi^{I_1 \dots I_3}{}_J \oplus \phi_{I_1 \dots I_3}{}^J \oplus \phi_{(I_1 I_2)} \oplus \phi^{(I_1 I_2)} \quad (7.10)$$

where $I, J, \dots = 1, 2, \dots, 8$. The reader can find a detailed account of how the next to top fields arise from the eleven dimensional fields by dimensional reduction in reference [37].

In particular, let us look for a theory that has a $SO(8)$ gauging of the E_7 symmetry. This is achieved if we take the next to top field $A_{a_1 a_2 a_3 \bullet}$ to be a singlet under $SO(8)$. Looking at the above representations of $SL(8)$ of equation (7.9), and decomposing them into $SO(8)$ representations, we see that there are only two singlets, one in the 36 and the other in the $\overline{36}$. The fields of the 36-dimensional representation arises from the eleven dimensional fields A_3 and $A_{9,6}$ which is consistent with the known eleven dimensional origin of this four dimensional gauged supergravity that arises from dimensional reduction on a seven sphere [37]. The $SO(8)$ singlet in the other 36-dimensional representation arises from the eleven dimensional fields $A_{10,1,1}$ and $A_{10,7,7}$ which shows that this four dimensional gauged supergravity has no eleven-dimensional supergravity origin, but of course it does have an eleven dimensional origin in the $E_{11} \otimes_s l_1$ non-linear realisation [37].

While it is clear from the above discussion that the $E_{11} \otimes_s l_1$ non-linear realisation does contain all the gauged supergravities, it would be interesting to show in detail how the dynamical equations that encode their origin follow from the non-linear realisation.

The spacetime contained in the non-linear realisation also contains an infinite number of coordinates whose detailed form is only known at low levels. However, just as for the fields one can find all the form coordinates, that is, coordinates that have one block of antisymmetrised indices. The result for the corresponding generators in the l_1 representation are given in the following table. [27, 54, 26].

Table 2. The form generators in the l_1 representation in D dimensions

D	G	Z	Z^a	$Z^{a_1 a_2}$	$Z^{a_1 \dots a_3}$	$Z^{a_1 \dots a_4}$	$Z^{a_1 \dots a_5}$	$Z^{a_1 \dots a_6}$	$Z^{a_1 \dots a_7}$
8	$SL(3) \otimes SL(2)$	$(\mathbf{3}, \mathbf{2})$	$(\mathbf{3}, \mathbf{1})$	$(\mathbf{1}, \mathbf{2})$	$(\mathbf{3}, \mathbf{1})$	$(\mathbf{3}, \mathbf{2})$	$(\mathbf{1}, \mathbf{3})$ $(\mathbf{8}, \mathbf{1})$ $(\mathbf{1}, \mathbf{1})$	$(\mathbf{3}, \mathbf{2})$ $(\mathbf{6}, \mathbf{2})$	$(\mathbf{6}, \mathbf{1})$ $(\mathbf{18}, \mathbf{1})$ $(\mathbf{3}, \mathbf{1})$ $(\mathbf{6}, \mathbf{1})$ $(\mathbf{3}, \mathbf{3})$
7	$SL(5)$	$\mathbf{10}$	$\bar{\mathbf{5}}$	$\mathbf{5}$	$\bar{\mathbf{10}}$	$\mathbf{24}$ $\mathbf{1}$	$\mathbf{40}$ $\mathbf{15}$ $\mathbf{10}$	$\mathbf{70}$ $\mathbf{50}$ $\mathbf{45}$ $\mathbf{5}$	- - - -
6	$SO(5, 5)$	$\bar{\mathbf{16}}$	$\mathbf{10}$	$\mathbf{16}$	$\mathbf{45}$ $\mathbf{1}$	$\bar{\mathbf{144}}$ $\mathbf{16}$	$\mathbf{320}$ $\mathbf{126}$ $\mathbf{120}$	- - -	- - -
5	E_6	$\mathbf{27}$	$\mathbf{27}$	$\mathbf{78}$ $\mathbf{1}$	$\mathbf{351}$ $\bar{\mathbf{27}}$	$\mathbf{1728}$ $\mathbf{351}$ $\mathbf{27}$	- - -	- - -	- - -
4	E_7	$\mathbf{56}$	$\mathbf{133}$ $\mathbf{1}$	$\mathbf{912}$ $\mathbf{56}$	$\mathbf{8645}$ $\mathbf{1539}$ $\mathbf{133}$ $\mathbf{1}$	- - - -	- - - -	- - - -	- - - -
3	E_8	$\mathbf{248}$ $\mathbf{1}$	$\mathbf{3875}$ $\mathbf{248}$ $\mathbf{1}$	$\mathbf{147250}$ $\mathbf{30380}$ $\mathbf{3875}$ $\mathbf{248}$ $\mathbf{1}$	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -

From the above table for the generators in the l_1 representation we can read off the coordinates in the spacetime that occurs in the non-linear realisation. At level zero we find the coordinates of the spacetime in D dimensions that we are familiar with. However, at level one we find coordinates which are scalars under the $SL(D)$ transformations of our usual spacetime, and so also Lorentz transformations, but belong to non-trivial representations of E_{11-D} . In particular, they belong to the

$$10, \quad \bar{\mathbf{16}}, \quad \bar{\mathbf{27}}, \quad 56, \quad \text{and} \quad 248 \oplus 1, \quad \text{of} \quad SL(5), \quad SO(5, 5), \quad E_6, \quad E_7 \quad \text{and} \quad E_8 \quad (1.2)$$

for $D = 7, 6, 5, 4$ and 3 dimensions respectively [27,54].

The coordinates play an essential role in the derivation of the equations of motion and one can not find invariant equations without them. We see from equation (6.18) that if the first (l_1) index of the Cartan form is of level one, that is, $G^{a_1 a_2}{}_{\bullet}$, then it varies into a Cartan form with a first index that is a usual spacetime index and so it contains derivatives with respect to the usual coordinates. Hence, the terms in the equations of motion with derivatives with respect to the usual derivatives and the higher level derivatives mix. As we explained above, to find the gauged supergravities in the non-linear realisation one had to take some next to top forms to be non-zero but one also has to take the fields to depend on the higher level coordinates in a non-trivial way [38]. Nonetheless, the physical role that the higher level coordinates play is not at all well understood. However, the very fact that the final truncated equations of motion are precisely those of the maximal supergravity theories and that the higher level coordinates are essential to find this result suggests that they play an important role in a way we have yet to understand. Given the unfamiliar nature of the higher level coordinates rather than give in to the temptation to invent mathematical tricks to try to eliminate them it may be better to try to find their underlying physical meaning.

8 Discussion

We have reviewed the theory of non-linear realisations and explained how it leads to dynamical equations of motion. We also have recalled how non-linear realisations played a key role in the introduction of symmetry and spontaneous symmetry breaking into particle physics. The theory of Kac-Moody algebras was briefly discussed as well as the construction of the Kac-Moody algebra E_{11} together with its vector representation l_1 . The non-linear realisation of $E_{11} \otimes_s l_1$ was constructed and the dynamics that it implies was derived. It leads to an E_{11} invariant field theory that has an infinite number of fields which depend on a spacetime that has an infinite number of coordinates. However, the uniquely determined dynamics agrees precisely with the equations of motion of the eleven dimensional supergravity theory when we restrict the fields to be those at lowest levels and the coordinates to be just those of our usual spacetime.

The dynamics in eleven dimensions was derived by taking a decomposition of E_{11} into its $GL(11)$ subgroup. However, by taking decompositions of E_{11} into different subalgebras we found all the maximal supergravity theories in ten and less dimensions. Although the detailed calculations have only been carried out in five dimensions, it is inevitable that the dynamics of the non-linear realisation of $E_{11} \otimes_s l_1$ in the different decompositions will agree with the equations of motion of the corresponding supergravity theories, in the same sense as just mentioned above. We also explained how the maximal gauged supergravities are automatically included in the $E_{11} \otimes_s l_1$ non-linear realisation. As result the $E_{11} \otimes_s l_1$ non-linear realisation is a unified theory in the sense that it contains all the maximal supergravities. It also follows from the way the different theories arise from the different decompositions that all the theories derived from the non-linear realisation are completely equivalent in that the coordinates and fields are just rearranged from one theory to another according to the different decompositions of $E_{11} \otimes_s l_1$ being used. We note that eleven dimensions does not play the preferred role as it does in M theory as all the theories are on an equal footing.

The maximal type II supergravity theories were thought to be the complete low energy effective actions for the type II superstrings. However, as the $E_{11} \otimes_s l_1$ non-linear realisation contains all these theories in one unified structure it is difficult not to believe that the conjecture of [23,24], namely that the $E_{11} \otimes_s l_1$ non-linear realisation is the low energy effective action for the type II superstrings. This theory contains many effects which are beyond those found in the supergravity theories and it will be very interesting to find out in detail what these effects are. Indeed this work provides a starting point from which to more systematically consider what is the underlying theory of strings and branes.

One obvious question is whether the higher level fields lead to additional degrees of freedom beyond those found at low levels which are just those of the maximal supergravity theories. While the answer to this question is not known for sure it is likely that this is not the case. As one examines the fields at levels just above the supergravity fields one does not find fields that lead to new degrees of freedom. Also, in eleven dimensions, all fields that do not have blocks of ten or eleven antisymmetrised indices have been classified [51]. One finds an infinite number of such fields, however, they are the fields of the supergravity theories plus fields that are dual to these fields. One expects that these dual fields satisfy first order duality relations and so not lead to any new degrees of freedom. Thus if one adopted a light-cone description which takes into account only objects which carry indices ranging over nine different values then one might, perhaps naively, expect that only the above fields would lead to dynamical degrees of freedom. As a result one expects to find only the usual degrees of freedom of the eleven dimensional supergravity theories. A possible exception is the dependence of the fields on the additional coordinates which could lead to new degrees of freedom as it does in higher spin theories [56]. The same conclusion applies to all the maximal super gravity theories.

As we have explained a non-linear realisation provides a very direct path from the algebra used in the non-linear realisation to the dynamics. In the case of the $E_{11} \otimes_s l_1$ non-linear realisation it provides a direct path from the E_{11} Dynkin diagram to the equations of motion of the maximal supergravities. We note that the dynamics of this non-linear realisation is uniquely determined, at least at low levels. The only assumptions we make are that we use the vector representation of E_{11} to build the semi-direct product algebra and that we require the smallest number space time derivatives which leads to non-trivial dynamics. It is amusing to note that one can uniquely derive Einstein's theory of general relativity in this way, that is, it is contained in this sense in the Dynkin diagram of E_{11} .

In this review we have focused entirely on the bosonic sector of the supergravity fields. One can introduce fermions as fields that transform under $I_c(E_{11})$ [57] following a similar procedure [58] to that carried out in the context of the E_{10} approach.

The symmetries of the non-linear realisation do not include the local symmetries of gauge and general coordinate transformations. However, the equations of motion that follow from the non-linear realisation are unique and they turn out to be general coordinate and gauge invariant. It would be interesting to see if this phenomenon persists at higher levels and why it is that these local symmetries arise in this way.

Although the coordinates beyond those of the usual spacetime must be truncated out of the equations of the $E_{11} \otimes_s l_1$ non-linear realisation to find the equations of motion we are used to, they play an essential role in the way the equations of motion were derived.

Indeed they are crucial for the E_{11} symmetry and one could not derive these equations without them. We should think of these extra coordinates as leading to physical effects, indeed they are required for the gauged supergravities. It is very unlikely that our usual notion of spacetime survives in a fundamental theory of physics and in particular in the underlying theory of strings and branes. One can think of the infinite dimensional spacetime that appears in the $E_{11} \otimes_s l_1$ non-linear realisation as a kind of low energy effect theory of spacetime that represents the properties of spacetime before it is replaced by more fundamental degrees of freedom. This can be thought of as analogous to the low energy effective actions which do not contain the fields that correspond to all the degrees of freedom in the underlying theory but only the fields corresponding to degrees of freedom which have a low mass compared to the scale being considered. The problem of how to eliminate all the higher level coordinates in the applications we are used to is a problem whose resolution demands a physical as well as a mathematical idea. Truncating the coordinates breaks the E_{11} symmetry, however when one better understands the role of the extra coordinates this breaking may appear as some kind of spontaneous rather than explicit symmetry breaking. As we recalled from the history of particle physics, one can not hope to solve all the problems in one go.

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